Massachusetts Institute of Technology | Handout 28
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Problem Set 8 Solutions

Problem 1. Undirected graphs

In an undirected graph on n vertices, an edge is an unordered pair of vertices, so there are $\binom{n}{2}$ possible edges. Each possible edge is either present or absent, so there are $2^{\binom{n}{2}}$ possible undirected graphs on n vertices.

Problem 2. Distinguishable dice

Five ordinary distinguishable six-sided dice can land in $6^5 = 7776$ ways. If they land so that no pair shows up, there are $6 \times 5 \times 4 \times 3 \times 2 = 720$ possibilities. To get a full house, there are $\binom{5}{2}$ ways to pick which dice are the pair and which are the triple, and 6×5 ways to choose the values of the pair and triple: $\binom{5}{2} \times 6 \times 5 = 300$.

Problem 3. Coefficient

By the binomial theorem, the typical term in this expansion is $\binom{100}{j}x^{100-j}(1/x)^j$, which simplifies to $\binom{100}{j}x^{100-2j}$. As j runs from 0 to 100, the exponent runs from 100 down to -100 in decrements of 2. If we let k denote the exponent, then solving $k = 100 - 2j$ for j, we obtain $j = (100 - k)/2$. Thus the values of k for which x^k appears in this expansion are $-100, -98, \ldots, -2, 0, 2, 4, \ldots, 100$, and for such values of k the coefficient is $\binom{100}{\frac{100-k}{2}}$.

Problem 4. Filenames

(a) $52^4 = 7311616$.

(b) $52^4 \div 2^4 = 456976 = 26^4$.

(c) $\sum_{i=1}^{4} 52^i = (52^5 - 52)/(52-1) = 7454980$; $\sum_{i=1}^{4} 26^i = (26^5 - 26)/(26-1) = 475254$. The division rule can be applied only to the individual terms of the summation.

Problem 5. Balls and urns

(a) $3 \cdot 3 \cdot 3 \cdot 3 = 81$.

(b) 14. We can put either all 4 balls in one urn, 3 in one, 2 in one and 2 in another, or 2 in one and 1 in each of the other two. The balls are distinguishable, so there are $\binom{4}{3}$ ways of putting 3 in one. There are $\binom{4}{2}$ ways of putting 2 in one and 1 in each of the other two. There are $\binom{4}{2}/2$ of putting 2 in one and 2 in another. So the total number of ways is

$$1 + \binom{4}{3} + \frac{1}{2}\binom{4}{2} + \binom{4}{2} = 14.$$

(c) 15. Create 6 slots and choose 2 to contain bars leaving r_1, r_2 and r_3 consecutive slots with stars with $r_1 + r_2 + r_3 = 4$. This corresponds to putting r_i balls in urn i. So the answer is $\binom{6}{2} = 15$.

(d) 4. This is the number of ways to partition a multiset $\{1, 1, 1, 1\}$ of 4 indistinguishable objects into 3 (possibly empty) subsets. We can either put $\{1, 1, 1, 1, \}, \emptyset, \emptyset$, or $\{1, 1, 1\}, \{1\}, \emptyset$, or $\{1, 1\}, \{1, 1\}, \emptyset$, or $\{1, 1\}, \{1\}, \{1\}$. The ordering in which we list the subsets doesn't matter because the urns are indistinguishable.

Problem 6. Integers not divisible

Here we use inclusion-exclusion to count the number of integers from 1 to 1000 not divisible by 6, 10, or 15. Make a table, and add the numbers with appropriate signs:

divisible by each of	least common multiple of divisors	number of integers from 1 to 1000	include/exclude +/−
	1	1000	+
6	6	166	−
10	10	100	−
15	15	66	−
6, 10	30	33	+
6, 15	30	33	+
10, 15	30	33	+
6, 10, 15	30	33	−

These add up to 734.

Problem 7. Binomial coefficient identity

(a)

$$\begin{aligned} x^{-n}(1+x)^n x^n (1+x^{-1})^n &= x^{-n}(1+x)^n(x+1)^n \\ (1+x)^n(1+x^{-1})^n &= x^{-n}(1+x)^{2n} \end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}x^i\sum_{j=0}^{n}\binom{n}{j}x^{-j} &= x^{-n}\sum_{i=0}^{2n}\binom{2n}{i}x^i \\ \sum_{i=0}^{n}\binom{n}{i}^2 &= \binom{2n}{n}\end{aligned}$$

by equating the coefficients of x^0.

(b) Suppose we have $2n$ objects, n of which are red, and n of which are blue. How many ways can we pick r red ones and $n-r$ blue ones? Clearly this is

$$\binom{n}{r}\binom{n}{n-r} = \binom{n}{r}^2.$$

How many ways can we choose n objects total, regardless of how many are of each color? This is just

$$\sum_{r=0}^{n}\binom{n}{r}^2.$$

But there are $\binom{2n}{n}$ ways to choose n objects, so these must be the same number.

Problem 8. Inclusion-Exclusion Principle

(a) The base case is $n = 2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. Look at a situation with $n+1$ sets, and temporarily consider $A_n \cup A_{n+1}$ as one set. Then by the inductive hypothesis we have

$$\begin{aligned}|A_1 \cup \ldots \cup A_{n+1}| &= \sum_{i<n}|A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n}|A_i \cap A_j| \\ &- \sum_{i<n}|A_i \cap (A_n \cup A_{n+1})| + \ldots + (-1)^n|A_1 \cap \ldots \cap A_{n-1} \cap (A_n \cup A_{n+1})|\end{aligned}$$

Next we apply the distributive law to each term on the right involving $A_n \cup A_{n+1}$, giving us

$$\sum|(A_{i_1} \cap \ldots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum|(A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \ldots \cap A_{i_m} \cap A_{n+1})|.$$

Now we apply the base case to rewrite each of these terms as

$$\sum|A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n| + \sum|A_{i_1} \cap \ldots \cap A_{i_m} \cap A_{n+1}| - \sum|A_{i_1} \cap \ldots \cap A_{i_m} \cap A_n \cap A_{n+1}|,$$

which gives us precisely the summation we want.

(b) Square-free numbers are those not divisible by the square of a prime. We count them as follows:

$$99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61.$$