Problem Set 7 Solutions

Problem 1. **(a)** Let s_n be the number of such sequences. A string ending in n must consist of a string ending in something less than n, followed by an n as the last term. Therefore the recurrence relation is $s_n = s_{n-1} + s_{n-2} + \ldots + s_2 + s_1$. Here is another approach, with a more compact form of the answer. A sequence ending in n is either a sequence ending in $n-1$, followed by n (and there are clearly s_{n-1} of these), or else it does not contain $n-1$ as a term at all, in which case it is *identical* to a sequence ending in $n-1$ in which the $n-1$ has been replaced by an n (and clearly there are s_{n-1} of these as well). Therefore $s_n = 2s_{n-1}$. Finally we notice that we can derive the second form from the first (or vice versa) algebraically (for example, $s_4 = 2s_3 = s_3 + s_3 = s_3 + s_2 + s_2 = s_3 + s_2 + s_1$).

(b) We need two initial conditions if we use the second formulation above, $s_1 = 1$ and $s_2 = 1$ (otherwise, our argument is invalid, because the first and last terms are the same). There is one sequence ending in 1, namely the sequence with just this 1 in it, and there is only the sequence $1, 2$ ending in 2. If we use the first formulation above, then we can get by with just the initial condition $s_1 = 1$.

(c) Clearly the solution to this recurrence relation and initial condition is $s_n = 2^{n-2}$ for all $n \geq 2$.

Problem 2. Using the Akra-Bazzi generalization of the master theorem, we have that for T, $k = 1, a_1 = 7, b_1 = 1/2, g(n) = n^2$ and for T', $k = 1, a_1 = a, b_1 = 1/4, g(n) = n^2$. In both cases, $g'(n) = 2n$ which is polynomially bounded. In the first case (for T), we have that $7 \cdot (1/2)^p = 1$ or that $p = \log_2 7$. In the second case (for T'), we have that $a \cdot (1/4)^p = 1$ or that $a \cdot (1/2)^{2p} = 1$, or that $p = \frac{1}{2}\log_2 a$. This gives

$$T(n) = \Theta(n^{\log_2 7}(1 + \int_1^n \frac{u^2}{u^{\log_2 7+1}} du)) = \Theta(n^{\log_2 7})$$

and

$$T'(n) = \Theta(n^{(\log_2 a)/2}(1 + \int_1^n \frac{u^2}{u^{(\log_2 a)/2+1}} du)) = \Theta(n^{(\log_2 a)/2}(1 + \int_1^n u^{1-(\log_2 a)/2} du)),$$

so that if $(\log_2 a)/2 > 2$, we have that $T'(n) = \Theta(n^{(\log_2 a)/2})$ and if $(\log_2 a)/2 = 2$, we have that $T'(n) = \Theta(n^{(\log_2 a)/2} \log n) = \Theta(n^2 \log n)$ and if $(\log_2 a)/2 < 2$, we have that $T'(n) = \Theta(n^2)$.

This means that for A' to be asymptotically faster than A, we need to have $\frac{1}{2}\log_2 a < \log_2 7$, or that $a < 7^2$, so that the largest integer value of a is 48.

Problem 3. Note that $\Theta(\log_a n) = \Theta(\log_b n)$, so we will leave out the bases for the logarithms when we are using asymtotic notation.

(a) $a_1 = 3$, $b_1 = 1/3$, $h_1(n) = \lfloor n/3 \rfloor - n/3$, $g(n) = n$, $p = 1$,

$$T(n) = \Theta(n(1 + \int_1^n \frac{u}{u^2} du)) = \Theta(n \log n).$$

(b) $a_1 = 4$, $b_1 = 1/3$, $h_1(n) = \lfloor n/3 \rfloor - n/3$, $g(n) = n^2$, $p = \log_3 4$,

$$T(n) = \Theta(n^{\log_3 4}(1 + \int_1^n \frac{u^2}{u^{\log_3 4+1}} du)) = \Theta(n^{\log_3 4}(1 + \int_1^n u^{1-\log_3 4} du)) = \Theta(n^2).$$

(c) $a_1 = 3$, $b_1 = 1/4$, $h_1(n) = \lfloor n/4 \rfloor - n/4$, $g(n) = n$, $p = \frac{1}{2}\log_2 3$,

$$T(n) = \Theta(n^{(\log_2 3)/2}(1+\int_1^n \frac{u}{u^{(\log_2 3)/2+1}} du)) = \Theta(n^{(\log_2 3)/2}(1+\int_1^n u^{-(\log_2 3)/2} du)) = \Theta(n).$$

(d) $a_1 = 1$, $a_2 = 1$, $b_1 = 1/4$, $b_2 = 1/3$, $h_1(n) = \lfloor n/4 \rfloor - n/4$, $h_2(n) = \lfloor n/3 \rfloor - n/3$, $g(n) = n$, $p = 0.5605$ (the exact value does not matter, it turns out that for this case, we only need to know that $p < 1$),

$$T(n) = \Theta(n^p(1 + \int_1^n \frac{u}{u^{p+1}} du)) = \Theta(n^p(1 + \int_1^n u^{-p} du)) = \Theta(n).$$

(e) $a_1 = 1$, $a_2 = 1$, $b_1 = 1/4$, $b_2 = 3/4$, $h_1(n) = \lceil n/4 \rceil - n/4$, $h_2(n) = \lfloor 3n/4 \rfloor - 3n/4$, $g(n) = n$, $p = 1$,

$$T(n) = \Theta(n(1 + \int_1^n \frac{u}{u^2} du)) = \Theta(n \log n).$$

(f) $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = 0$, $g(n) = \sqrt{n}$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{\sqrt{u}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$

(g) $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = 1$, $g(n) = \sqrt{n}$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{\sqrt{u}}{u^{3/2}} du)) = \Theta(n^{1/2} \log n).$$

(h) $a_1 = 2$, $b_1 = 1/4$, $h_1(n) = \sqrt{n}$, $g(n) = 1$, $p = 1/2$,

$$T(n) = \Theta(n^{1/2}(1 + \int_1^n \frac{1}{u^{3/2}} du)) = \Theta(n^{1/2}).$$

(i) Write $n = 3^m$ and $S(m) = T(3^m)$. Then the recurrence becomes

$$S(m) = 3S(\frac{m}{3}) + m,$$

for which $a_1 = 3$, $b_1 = 1/3$, $h_1(m) = 0$, $g(m) = m$, $p = 1$,

$$S(m) = \Theta(m(1 + \int_1^m \frac{u}{u^2} du)) = \Theta(m \log m),$$

so that

$$T(n) = S(\log n) = \Theta(\log n \log \log n).$$

(j) Write $n = e^m$ and $S(m) = T(e^m)$. Then the recurrence becomes

$$S(m) = \sqrt{e} S(\frac{m}{e}) + m,$$

for which $a_1 = \sqrt{e}$, $b_1 = 1/e$, $h_1(m) = 0$, $g(m) = m$, $p = 1/2$,

$$S(m) = \Theta(m^{1/2}(1 + \int_1^m \frac{u}{u^{3/2}} du)) = \Theta(m),$$

so that

$$T(n) = S(\log n) = \Theta(\log n).$$

Problem 4. All logarithms in this question are base 2. We will present exact solutions to both recurrences below. The claims were obtained by "plug-and-chug". Only the inductive proof will be shown below. We will let c_1 and c_2 be constants such that $c_1 \leq T(n) \leq c_2$ for $n \leq 2$.

(a) We claim that $\lceil \log \log n \rceil + c_1 \leq T(n) \leq \lceil \log \log n \rceil + c_2$, for $n \leq 2$. This can be proven by induction. The base case follows immediately, since $\log \log n = 0$ for $n \leq 2$. For $n > 2$, we have $T(n) = T(\sqrt{n}) + 1 \geq \lceil \log \log \sqrt{n} \rceil + c_1 + 1$ by inductive hypothesis. This simplifies to $T(n) \geq \lceil \log(\frac{1}{2} \log n) \rceil + c_1 + 1 = \lceil \log 1/2 + \log \log n \rceil + c_1 + 1 = \lceil \log \log n - 1 \rceil + c_1 + 1 = \lceil \log \log n \rceil + c_1$. On the other side, we have $T(n) = T(\sqrt{n}) + 1 \leq \lceil \log \log \sqrt{n} \rceil + c_2 + 1$ by inductive hypothesis. This simplifies to $T(n) \leq \lceil \log(\frac{1}{2} \log n) \rceil + c_2 + 1 = \lceil \log 1/2 + \log \log n \rceil + c_2 + 1 = \lceil \log \log n - 1 \rceil + c_2 + 1 = \lceil \log \log n \rceil + c_2$, completing the induction.

So $T(n) = \Theta(\log \log n)$ is asymptotically tight.

(b) We claim that $n \lceil \log \log n \rceil + c_1 n^{1-2^{-\lceil \log \log n \rceil}} \leq T(n) \leq n \lceil \log \log n \rceil + c_2 n^{1-2^{-\lceil \log \log n \rceil}}$. This can be proven by induction. For the base case, we can easily verify that our claim implies that $c_1 \leq T(n) \leq c_2$, since $\lceil \log \log n \rceil = 0$, which is true by the definition of c_1 and c_2.

For $n > 2$, we have that $T(n) = \sqrt{n}T(\sqrt{n}) + n$ which is at least $\sqrt{n}[\sqrt{n}\lceil \log\log \sqrt{n}\rceil + c_1(\sqrt{n})^{1-2^{-\lceil \log\log \sqrt{n}\rceil}}] + n$ by inductive hypothesis. Simplifying gives us

$$\begin{aligned} T(n) &\geq \sqrt{n}[\sqrt{n}\lceil \log\log n - 1\rceil + c_1 n^{(\frac{1}{2})(1-2^{-\lceil \log\log n-1\rceil})}] + n \\ &= n\lceil \log\log n\rceil - n + c_1 n^{\frac{1}{2}+\frac{1}{2}-2^{-\lceil \log\log n-1\rceil-1}} + n \\ &= n\lceil \log\log n\rceil + c_1 n^{1-2^{-\lceil \log\log n\rceil}}. \end{aligned}$$

The other side is essentially the same. This time, we have that $T(n) \leq \sqrt{n}[\sqrt{n}\lceil \log\log \sqrt{n}\rceil + c_2(\sqrt{n})^{1-2^{-\lceil \log\log \sqrt{n}\rceil}}] + n$ by inductive hypothesis. Simplifying gives us

$$\begin{aligned} T(n) &\leq \sqrt{n}[\sqrt{n}\lceil \log\log n - 1\rceil + c_2 n^{(\frac{1}{2})(1-2^{-\lceil \log\log n-1\rceil})}] + n \\ &= n\lceil \log\log n\rceil - n + c_2 n^{\frac{1}{2}+\frac{1}{2}-2^{-\lceil \log\log n-1\rceil-1}} + n \\ &= n\lceil \log\log n\rceil + c_2 n^{1-2^{-\lceil \log\log n\rceil}}, \end{aligned}$$

completing the induction. The asymptotically tight bounds are therefore $T(n) = \Theta(n \log\log n)$.

Problem 5. $T(n) - (n-1)T(n-1) = \sum_{i=1}^{n-2} iT(i) = T(n-1) \implies T(n) = (n-1)T(n-1) + T(n-1) \implies T(n) = nT(n-1)$.

$T(1) = 1,\ T(2) = 1 \cdot T(1) = 1,$ and $T(3) = 2 \cdot T(2) + 1 \cdot T(1) = 3$. So we have:

$$T(n) = \begin{cases} 1 & \text{for } n = 1 \\ n!/2 & \text{for } n \geq 2 \end{cases}$$

Verify by induction:

Proof. Base Case: $n = 2 \implies T(2) = \sum_{i=1}^{1} iT(i) = 1 \cdot T(1) = 1 = 2!/2$.

Inductive Step:

$$\begin{aligned} T(n+1) &= (n+1)T(n) \\ &= (n+1)\ n!/2 \qquad \text{by the induction hypothesis} \\ &= (n+1)!/2 \end{aligned}$$

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Problem 6. First solve the general solution of the homogeneous recurrence: $T(n) = 4T(n-1) - 4T(n-2)$. Its characteristic equation is: $\alpha^2 - 4\alpha + 4 = 0 \implies (\alpha-2)^2 = 0 \implies 2$ is a root with multiplicity 2. Solutions are all linear combinations of 2^n and $n2^n$, i.e. $A2^n + Bn2^n$

for $A, B \in \mathbb{R}$. Find any particular solution to the nonhomogeneous recurrence (ignoring initial conditions). For a polynomial "drive", try polynomial of equal degree (higher if 1 is a root of characteristic equation). Guess $T(n) = an^2 + bn + c$, and solve for a, b, and c satisfying the recurrence:

$$\begin{aligned}
an^2 + bn + c &= 4(a(n-1)^2 + b(n-1) + c) - 4(a(n-2)^2 + b(n-2) + c) + n^2 \\
&= 4((n-1)^2 - (n-2)^2) + 4b((n-1) - (n-2)) + 4c(1-1) + n^2 \\
&= 4a(2n-3) + 4b + n^2 \\
&= n^2 + 8an + 4b - 12c
\end{aligned}$$

which implies that $a = 1,\ b = 8a = 8,\ c = 4b - 12a = 32 - 12 = 20.$

So general solution to inhomogeneous recurrence is $A2^n + Bn2^n + n^2 + 8n + 20$ for $A, B \in \mathbb{R}$.

Find values for A, B meeting initial conditions:

$0 = T(0) = A + 20 \Longrightarrow A = -20.$

$1 = T(1) = 2A + 2B + 1 + 8 + 20 = 2B - 40 + 29 \Longrightarrow B = 1 + 40 - 29 = 12 \Longrightarrow B = 6.$

Solution: $T(n) = (6n - 20)2^n + n^2 + 8n + 20.$

Check answers for small values of n (to avoid arithmetic and algebraic errors):

$T(0) = -20 + 20 = 0$

$T(1) = -14 \cdot 2 + 1 + 8 + 20 = -28 + 29 = 1$

$T(2) = -8 \cdot 4 + 4 + 16 + 20 = -32 + 40 = 8 = 4 \cdot 1 - 4 \cdot 0 + 2^2.$

Problem 7. **(a)** $f(3) = f(2) + f(1) + 2^1 = 5$

$f(4) = f(3) + f(2) + 2^2 = 11$

$f(5) = f(4) + f(3) + 2^3 = 11 + 5 + 8 = 24$

(b) $f(n) = f(n-1) + f(n-2) + 2^{n-2}$

(c) Solve homogeneous recurrence: It is just the Fibonacci recurrence, where roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$ (of course we could solve the characteristic polynomial as well).

Find particular solution: for exponential "drive", try exponential with same base. Guess $f(n) = A2^n \Longrightarrow A2^n = A2^{n-1} + A2^{n-2} + 2^{n-2} \Longrightarrow 4A - 2A - A = 1 \Longrightarrow A = 1.$

General solution to recurrence is $f(n) = B(\frac{1+\sqrt{5}}{2}) + C(\frac{1-\sqrt{5}}{2}) + 2^n$, for $B, C \in \mathbb{R}$.

Solve for B and C meeting initial conditions:

$0 = f(0) = B + C + 1 \Longrightarrow C = -(B+1).$

$1 = f(1) = B(\frac{1+\sqrt{5}}{2}) + C(\frac{1-\sqrt{5}}{2}) + 2 = B(\frac{1+\sqrt{5}}{2}) - (B+1)(\frac{1-\sqrt{5}}{2}) + 2 = B(\sqrt{5}) - \frac{1-\sqrt{5}}{2} + 2 \Longrightarrow B = \frac{1}{\sqrt{5}}(-1 + \frac{1-\sqrt{5}}{2}) = \frac{-1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2}),$ and

$C = -(B+1) = \frac{1-\sqrt{5}}{2\sqrt{5}} = \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})$.
So:

$$\begin{aligned} f(n) &= 2^n - \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} + \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1} \\ &= 2^n - (\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1}) \\ &= 2^n - F_{n+1}, \quad \text{where} \quad F_{n+1} \quad \text{is the } n+1^{st} \text{ Fibonacci number.} \end{aligned}$$