Problem Set 8

Due: Start of Class, Tuesday, November 5, 1996.

Reading: Sections 4.1, 4.3, 5.4–5.5 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 1. **[9 points]**

How many edges are possible in an undirected graph on the set $V_n = \{1, 2, \ldots, n\}$ of vertices? How many undirected graphs are possible on V_n? Note that since the vertices are labeled and therefore distinguishable, $G = (V_3, \{(1,2)\})$ and $G' = (V_3, \{(1,3)\})$ are different graphs.

Problem 2. **[10 points]**

How many different ways can five ordinary six-sided distinguishable dice land? How many ways can they land so that no pair shows? How many ways can they land so that a full house shows, that is, a pair and a triple, but not five-of-a-kind? Provide numerical answers, but show your derivations.

Problem 3. **[10 points]** (Exercise 4.3-42 of Rosen)

Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k ia an integer.

Problem 4. **[15 points]**

A computer operating system requires file names to be exactly four letters long; uppercase and lowercase letters are distinct.

(a) How many possible file names are there?

(b) The operating system is changed so that uppercase and lowercase are no longer distinct. Figure out the new number of file names in two ways: applying the division rule to **(a)** and directly.

(c) The operating system is changed so that file names may be *up to* four letters long. Repeat **(a)** and **(b)**. Can the division rule still be used?

Problem 5. **[20 points]**

How many ways can 4 balls be placed in 3 urns if

(a) both the balls and the urns are distinguishable;

(b) the balls are distinguishable but the urns are not;

(c) the urns are distinguishable but the balls are not;

(d) neither the balls nor the urns are distinguishable.

Problem 6. [**10 points**]

How many integers in the range from 1 to 1000 are not divisible by any of the numbers 6, 10, and 15?

Problem 7. [**16 points**]

We wish to prove the identity

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n},$$

for $n \in \mathbb{N}$.

(a) Use the binomial theorem and the fact that

$$(1+x)^n(1+x^{-1})^n = x^{-n}(1+x)^{2n}$$

to prove the identity.

(b) Consider a set of $2n$ objects, of which n are red and n are blue. Use the fact that

$$\binom{n}{r}^2 = \binom{n}{r}\binom{n}{n-r}$$

to give a combinatorial proof of the identity.

Problem 8. (Inclusion-Exclusion principle and its application)

[**10 points**] (Exercises 5.4-22 and 5.5-6 of Rosen)

(a) Prove the principle of inclusion-exclusion using mathematical induction.

(b) An integer is called **squarefree** if it is not divisible by the square of a positive integer greater than 1. Find the number of squarefree positive integers less than 100.