Problem Set 6 Solutions

Problem 1. Present value of what the college loan officer offers

$$\begin{aligned}
&= 22000 + 22000/1.08 + 22000/(1.08)^2 + 22000/(1.08)^3 \\
&= 22000(1 - \frac{1}{(1.08)^4})/(1 - \frac{1}{1.08}) \\
&= 22000 * 3.577 \\
&= \$78,696.13
\end{aligned}$$

Present value of what you must pay in return

$$\begin{aligned}
&= 10000/(1.08)^4 + 10000/(1.08)^5 + \ldots + 10000/(1.08)^{15} \\
&= (10000/(1.08)^4)(1 - \frac{1}{(1.08)^{12}})/(1 - \frac{1}{1.08}) \\
&= \$59,823.82
\end{aligned}$$

This is too fair of a deal for you!

Problem 2. We first come up with a formula for $\sum_{i=1}^{n} i^2 x^i$ and then prove it using induction. From class notes we know that

$$\sum_{i=1}^{n} i x^i = \frac{x - (n+1)x^{n+1} + n x^{n+2}}{(1-x)^2}.$$

Differentiating (with respect to x) the above expression and multiplying with x we find that

$$\sum_{i=1}^{n} i^2 x^i = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

Proof by induction. Let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i^2 x^i = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

Base Case. $P(1)$ is true since

$$\sum_{i=1}^{1} i^2 x^i = x = \frac{x + x^2 - 4x^2 + 3x^3 - x^4}{(1-x)^3}.$$

Inductive Step. Assume $P(n)$ is true. Then

$$\begin{aligned}
\sum_{i=1}^{n} i^2 x^i + (n+1)^2 x^{n+1} &= \\
\sum_{i=1}^{n+1} i^2 x^i &= \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3} + (n+1)^2 x^{n+1} \\
&= \frac{x + x^2 - (n+2)^2 x^{n+2} + (2(n+1)^2 + 2(n+1) - 1)x^{n+3} - (n+1)^2 x^{n+4}}{(1-x)^3}
\end{aligned}$$

Thus $P(n+1)$ is true. Hence, $P(n)$ is true for all n.

Problem 3. We use the integral method for bounding the infinite sum. Since $f(k) = \frac{1}{k^{3/2}}$ is monotonically decreasing for $k > 1$, we have:

$$\begin{aligned}
\sum_{k=1}^{\infty} 1/k^{3/2} &= 1 + \sum_{k=2}^{\infty} 1/k^{3/2} \\
&\leq 1 + \int_1^{\infty} \frac{1}{x^{3/2}} dx \\
&= 1 + \left[\frac{-2}{\sqrt{x}}\right]_1^{\infty} \\
&= 1 + (0 + 2) \\
&= 3
\end{aligned}$$

Hence, $\sum_{k=1}^{\infty} 1/k^{3/2} \leq 3$.

Problem 4.

$$\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{ij}{2} x^{i/2} y^{j+i/2} &= \frac{1}{2} \cdot \left(\sum_{i=1}^{\infty} i(\sqrt{xy})^i\right) \cdot \left(\sum_{j=1}^{\infty} j y^j\right) \\
&= \frac{1}{2} \cdot \frac{\sqrt{xy}}{(1-\sqrt{xy})^2} \cdot \frac{y}{(1-y)^2} \\
&= \frac{\sqrt{x}\, y^{3/2}}{2(1-\sqrt{xy})^2(1-y)^2}
\end{aligned}$$

Problem 5. **(a)** Clearly $n! \leq n^n$, $\forall\ n \geq 1$. Therefore, $\log(n!) \leq \log(n^n) = n \log n$. Thus, $\log(n!) = O(n \log n)$.

(b) If we show $\log(n!) = \Omega(n \log n)$, then together with part (a) we have shown that $\log(n!) = \Theta(n \log n)$. Approximate $\log(n!)$ with *Stirling's formula* for $n!$ to get:

$$\begin{aligned}
\log(n!) &= \log(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{a_n}) \qquad \text{where } \tfrac{1}{12n+1} \leq a_n \leq \tfrac{1}{12n} \\
&= \log\sqrt{2\pi n} + \log n^n - \log e^n + \log e^{a_n} \\
&= \log\sqrt{2\pi n} + n\log n - n\log e + a_n \log e \\
&\geq n\log n - n\log e \\
&\geq \frac{1}{2} n \log n \qquad \text{since} \quad n\log e \leq \tfrac{n}{2}\log n \quad \text{for } n \geq e^2
\end{aligned}$$

Hence, $\log(n!) = \Omega(n \log n)$, which together with part (a) implies that $\log(n!) = \Theta(n \log n)$.

Problem 6. This does not follow. Let $f(x) = 2x$ and $g(x) = x$. Then $f(x) = O(g(x))$. Now $2^{f(x)} = 2^{2x} = 4^x$, and $2^{g(x)} = 2^x$, and $4^x \neq O(2^x)$. Indeed, $\frac{4^x}{2^x} = 2^x$, so the ratio grows without bound as x grows– it is not bounded by a constant.

Problem 7. **(a)** For $x > 2$ we certainly have $\lfloor x + 1/2 \rfloor \leq 2x$, and also $x \leq 2\lfloor x + 1/2 \rfloor$.

(b) For $x > 2$, $\log(x^2+1) \leq \log(2x^2) = 1 + 2\log x \leq 3\log x$ (recall that log means $\log_2$). On the other hand, since $x < x^2 + 1$ for all positive x, we have $\log x \leq \log(x^2+1)$.

(c) This follows from the fact that $\log_{10} x = C(\log_2 x)$, where $C = 1/\log_2 10$.