Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 21
October 22, 1996

Problem Set 7

Due: Start of Class, Tuesday, October 29.

Reading: Sections 5.1–5.3 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen, and *Notes on Better Master Theorems* by Tom Leighton.

Problem 1. **[15 points]**

Do exercise $5.1 - 16$ of Rosen.

Problem 2. **[10 points]**

The running time of an algorithm A is described by the recurrence $T(n) = 7T(n/2) + n^2$. A competing algorithm A' has a running time of $T'(n) = aT'(n/4) + n^2$. What is the largest integer value for a such that A' is asymptotically faster than A?

Problem 3. **[20 points]**

Use the formula given in class to find tight asymptotic bounds (Θ) for the following divide-and-conquer recurrences. For each recurrence, $T(1) = 1$ and $T(n) = \Theta(1)$ for all constant n. State the value of p you get for each recurrence (which can be left in the form of logs). Also, state the values of the $a_i, b_i,$ and $h_i(n)$ for each recurrence.

(a) $T(n) = 3T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$

(b) $T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n^2.$

(c) $T(n) = 3T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + n.$

(d) $T(n) = T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + n.$

(e) $T(n) = T\left(\left\lceil \frac{n}{4} \right\rceil\right) + T\left(\left\lfloor \frac{3n}{4} \right\rfloor\right) + n.$

(f) $T(n) = 2T(n/4) + \sqrt{n}.$

(g) $T(n) = 2T(n/4 + 1) + \sqrt{n}.$

(h) $T(n) = 2T(n/4 + \sqrt{n}) + 1.$

(i) $T(n) = 3T(n^{1/3}) + \log_3 n.$

(j) $T(n) = \sqrt{e}T(n^{1/e}) + \ln n.$

Problem 4. **[10 points]**

Give asymptotic upper and lower bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \leq 2$. Make your bounds as tight as possible, and justify your answers.

(a) $T(n) = T(\sqrt{n}) + 1$.

(b) $T(n) = \sqrt{n}T(\sqrt{n}) + n$.

Problem 5. **[10 points]**

Find an exact closed form solution for the recurrence

$$T(n) = \sum_{i=1}^{n-1} iT(i)$$

for $n > 1$ and $T(1) = 1$. (*Hint:* Find a simple formula for $T(n) - T(n-1)$.)

Problem 6. **[15 points]**

Find an exact closed form solution for the recurrence

$$T(n) = 4T(n-1) - 4T(n-2) + n^2$$

for $n > 1$, and $T(0) = 0$, $T(1) = 1$.

Problem 7. **[20 points]**

The graduate-student job problem made worse. Consider the following depressing scenario: Every year, each professor of computer science graduates one new graduate student who holds a postdoc for one year and then goes on to become a professor the following year. To compound the miseries of newly graduating students, in year $n \geq 2$, there are 2^{n-2} physicists who change careers to become computer science professors without going to graduate school. Let $f(n)$ denote the number of computer science professors after n years. Assume $f(0) = 0$ and $f(1) = 1$. It follows that $f(2) = 2$: in year 2, there are two CS professors, namely the one CS professor from year 1 and the one physicist transfer (there is also one postdoc who graduated in year 1 and will be professor in year 3).

(a) Compute $f(3)$, $f(4)$, and $f(5)$.

(b) Set up a recurrence relation obeyed by $f(n)$.

(c) Solve your recurrence to find a closed form for $f(n)$.