Quiz 1 Solutions

Problem 1. Prove or disprove

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case. $P(0)$ is true since

$$\sum_{i=0}^{0} i^2 = 0 = \frac{0(0+1)((2)(0)+1)}{6}.$$

Inductive step. Assume $P(n)$ is true. Then

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$\begin{aligned}\sum_{i=0}^{n+1} i^2 &= \sum_{i=0}^{n} i^2 + (n+1)^2 \\ &= \tfrac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1)\tfrac{2n^2+7n+6}{6} \\ &= \tfrac{(n+1)(n+2)(2n+3)}{6}\end{aligned}$$

Hence, $P(n+1)$ is true. Thus $P(n)$ is true for all n.

Problem 2. An infinite sequence of natural numbers $a_1, a_2, a_3, \ldots$ is defined in such a way that $a_1 = 14, a_2 = 35$, and each later a_i, $i \geq 3$, is obtained using one of the following rules:

Rule A Choose any two preceding elements of the sequence and define a_i to be their sum.

Rule B Choose any number $k \geq 1$ of preceding elements of the sequence, and define a_i to be their product divided by 7^{k-1}.

Use the well-ordering principle to prove that every element in the sequence is a multiple of 7. (*Hint: In case you have forgotten, the well-ordering principle says that every nonempty set of naturals has a smallest element.*)

Proof. Let $S = \{$all elements in the sequence that are not multiples of 7 $\}$.
Assume that S is nonempty. Then, by well-ordering, it has a smallest element a_n.
If a_n is obtained by rule A, then $a_n = a_i + a_j$ for some a_i and a_j smaller than a_n. But then a_i and a_j are multiples of 7 therefore a_n has to be a multiple of 7 also.
If a_n is obtained by rule B, then there are k smaller elements in the sequence ,that are multiples of 7, so we can write them in the form $a_k = 7n_k$. a_n is the product of the a_k divided by 7^{k-1}. Then we can write

$$a_n = (\prod_{i=1}^{k}(7n_i))/7^{k-1} = 7(\prod_{i=1}^{k} n_i)$$

Therefore a_n is also a multiple of 7. ∎

Problem 3. Using Euclid's algorithm, or otherwise, find an x such that

$$43x \equiv 1 \pmod{105}.$$

	105	43	−9	22
2	43	19	4	−9
2	19	5	−1	4
3	5	4	1	−1
1	4	1	0	1
1	1	0	1	0

So $x = 22$.

Problem 4. Let $f : U \to V$ be an arbitrary function and A, B be arbitrary subsets of U. Prove or disprove the following claims.

a) $f(A \cap B) = f(A) \cap f(B)$

Not True. Let $A = \{1\}, B = \{-1\}, f(x) = x^2$.

b) $f(A \cup B) = f(A) \cup f(B)$

True.

Proof. First we show $f(A \cup B) \subseteq f(A) \cup f(B)$.

$x \in f(A \cup B) \Rightarrow (f^{-1}(x) \subseteq A) \vee (f^{-1}(x) \subseteq B)$
$\Rightarrow (x \in f(A)) \vee (x \in f(B))$
$\Rightarrow x \in f(A) \cup f(B)$.

Now we show that $f(A) \cup f(B) \subseteq f(A \cup B)$.

$x \in f(A) \cup f(B) \Rightarrow (x \in f(A)) \vee (x \in f(B))$
$\Rightarrow (f^{-1}(x) \subseteq A) \vee (f^{-1}(x) \subseteq B)$
$\Rightarrow f^{-1}(x) \subseteq A \cup B$
$\Rightarrow x \subseteq f(A \cup B)$.

■

Problem 5. Which of the following functions are bijections? If a function is a bijection write down its inverse.

(a) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$

Injection $x \neq y \Rightarrow f(x) \neq f(y)$
Surjection $\forall r \in \mathbb{R}, \exists x, (x = r), s.t. f(x) = r.$
Inverse $f^{-1}(x) = x.$

(b) $f : \mathbb{R}^+ \to \mathbb{R}^+, f(x) = \frac{1}{x}$

Injection $x \neq y \Rightarrow \frac{1}{x} \neq \frac{1}{y}$
Surjection $\forall r \in \mathbb{R}, \exists x, (x = \frac{1}{r}), s.t. f(x) = r.$
Inverse $f^{-1}(x) = \frac{1}{x}.$

(c) $f : \mathbb{N} \to \mathbb{N}, f(x) = x^2$
This function is not a bijection because it is not a surjection. For example, there is no $n \in \mathbb{N}$ such that $n^2 = 3$.

(d) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$
Injection $x \neq y \Rightarrow x^3 \neq y^3$
Surjection $\forall r \in \mathbb{R}, \exists x, (x = r^{\frac{1}{3}}), s.t. f(x) = r.$
Inverse $f^{-1}(x) = x^{\frac{1}{3}}.$

Problem 6. Prove that among any set of 150 natural numbers, there must be three numbers, a, b, and c, such that all the pairwise differences, $a - b, a - c, b - c, b - a, c - a, c - b$, are multiples of 70.

Proof. Use Pigeonhole.
Pigeons The 150 numbers.
Holes The natural numbers mod 70.
There is some hole with $\left\lceil \frac{150}{70} \right\rceil = 3$ pigeons, that is, three numbers a, b, c that are equivalent modulo 70. Then their differences are equivalent to 0 modulo 70, therefore they are divisible by 70. ■

Problem 7. For each of the following sets, either use diagonalization to prove that the set is uncountable, or else use dovetailing or another technique to show that the set is countable. (If an answer to one part of the question follows from an answer you have already given to another part, then you need only indicate this – you don't need to give a direct proof of the part that follows.)

(a) The set E of all the even natural numbers.

This set is countable. The function $f(n) = 2n$ is a bijection from $\mathbb{N}$ to E.

(b) The set of all subsets of E.

This set is uncountable. The power set of any set has bigger cardinality than the original set.

(c) The set of all finite subsets of E.

Let $S_i =$ The set of all subsets of E with i elements.
Each of the S_i is countable and the set under question is the countable union of all the S_i, therefore it is countable.

(d) The set of all even-size finite subsets of E.

This set is countable, being a subset of the set in part (c).

Problem 8. The CS department has just completed a revamping of its curriculum. The following 10 courses are now required: 6.001, 6.002, 6.003, ..., 6.009, and 6.042.
In addition, for every i, j, 6.00i is a prerequisite of 6.00j or 6.0j, if $i < j$ and $gcd(i, j) > 1$.

(a) Write down all the elements of the *prerequisite* relation.

prerequisite = { (6.002,6.004) , (6.002,6.006) , (6.002,6.008) , (6.002,6.0042)
(6.003,6.006) , (6.003,6.009) , (6.003,6.042) , (6.004,6.006)
(6.004,6.008) , (6.004,6.042) , (6.006,6.008) , (6.006,6.042)
(6.007,6.042) , (6.008,6.042) , (6.009,6.042) , (6.006,6.009) }

(b) State whether this relation is reflexive, symmetric, transitive, antisymmetric. Is it an equivalence relation?

The relation is antisymmetric, but nothing else.

Problem continued on the next page.

(c) Now consider the *taken before* relation which is defined to be the reflexive and transitive closure of the *prerequisite* relation. Draw the Hasse Diagram for the *taken before* relation.

(d) The new regulations place no limit on how many courses a student can take in one term. What is the minimum number of terms required to complete all the required courses? Why?

The minimum number of terms is 5, which is the length of the critical path in the above Hasse diagram.

(e) Write down a maximum-size antichain in the partial order. What is the meaning of an antichain in terms of a student's schedule?

6.001, 6.002, 6.003, 6.005, 6.007.
An antichain is a list of subjects that a student can take at the same term.

Problem 9. A *Tournament Graph* is a directed graph such that if u and v are vertices then either (u, v) is an edge or (v, u) is an edge but not both. A *Directed Hamiltonian Path* is a simple path in a directed graph that contains every vertex exactly once.
Prove that every tournament graph has a directed Hamiltonian path. (*Hint: Use induction on the number of vertices.*)

Proof. We use induction on the number of vertices v in the graph. So let $P(v)$ be the predicate "Every tournament graph with v vertices has a hamiltonian path".

Base case $P(2)$. Let $V = \{x_1, x_2\}$. There are two possible tournament graphs, one in which (x_1, x_2) is the only edge, and one in which (x_2, x_1) is the only edge. In both cases, the existence of a hamiltonian path is clear (just follow the edge).

Inductive step Now suppose $P(v)$ is true. We want to establish $P(v + 1)$. So let $T = (V, E)$ be a tournament graph with $v + 1$ vertices. Pick some vertex x in V. Let T' be the subgraph of T with vertex set $V' = V - \{x\}$ and edge set E' consisting of all edges in E that connect vertices in V'. It is easily verified that this graph is a tournament with v vertices, and hence, since we assume $P(v)$ to be true, it contains a hamiltonian path $x_1, x_2, .., x_v$. There are three cases to consider.

1. Suppose first that (x, x_1) is in E. In this case, $x, x_1, .., x_v$ is a hamiltonian path in T.
2. Suppose that (x, x_1) is not in E, but (x_v, x) is. In this case, $x_1, ..., x_v, x$ is a hamiltonian path in T.
3. Suppose that neither (x, x_1) nor (x_v, x) is in E. This means that (x_1, x) and (x, x_v) are in E. Let S be the set of $i \in \{1, ..., v\}$ such that (x, x_i) is in E. By assumption, v is in S. Hence S is nonempty. By the well-ordering principle, there is a smallest element l of S. By assumption, $l > 1$. It follows that (x_{l-1}, x) and (x, x_l) are both in E. We can then construct the hamiltonian path $x_1, \ldots, x_{l-1}, x, x_l, .., x_v$ in T.

We conclude that $P(v + 1)$ is true, and hence $P(v)$ is true for all natural numbers by induction. ∎

Problem 10. Consider a state machine whose states consist of values for the following variables, all in $\mathbb{N}$:

x, y: initially arbitrary positive naturals
a: initially 1
r, s: initially equal to x and y, respectively

Its transitions are those of the following two kinds:

1. *even*
   Precondition:
   s is even

   $s > 0$

   Effect:
   $r := r^2$

   $s := \frac{s}{2}$

2. *odd*
   Precondition:
   s is odd

   Effect:
   $a := a \times r$

   $r := r^2$

   $s := \frac{s-1}{2}$

Use induction on the length of an execution to prove the following invariant of this state machine:

$$\text{If } s = 0 \text{ then } a = x^y.$$

(*Hint: You will need to strengthen this invariant in order to prove it by induction.*)

We are going to show the invariant $x^y = (r^s)a$

Base case: Initially $r = x, s = y, a = 1$, therefore $x^y = (r^s)a$.

Induction Step

- s is even
  $q^{'}.r = (q.r)^2$
  $q^{'}.s = (q.s)/2$
  $((q.r)^2)^{\frac{q.s}{2}}(q.a) = (q.r)^{q.s}(q.a) = x^y$

- s is odd
  $q^{'}.r = (q.r)^2$
  $q^{'}.s = (q.s - 1)/2$
  $q^{'}.a := (q.a)(q.r)$
  $((q.r)^2)^{\frac{q.s-1}{2}}(q.a)(q.r) = (q.r)^{q.s}(q.a) = x^y$

The induction is complete. If we set $s = 0$ in our invariant, we obtain $a = x^y$.