Problem Set 5 Solutions

Problem 1. **(a)** Clearly the numbers 27, 48, 60 and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.

(b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.

(c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

(d) There is no least element, since there is no number in the set that divides both 2 and 9.

(e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.

(f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

(g) We need to find numbers in the list that are divisors fo both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.

(h) Of the numbers we found in the previous parts, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

Problem 2. **(a)** First observe that $2e$ is the sum of the degrees of the vertices (each edge contributes a degree of 2 since it is incident to two vertices). But the total degree cannot be less than the minimum degree multiplied by the number of vertices. This means that $2e \geq mv$ or that $2e/v \geq m$.

(b) As before, observe that $2e$ is the sum of the degress of the vertices. Since the total degree cannot be greater than the maximum degree multiplied by the number of vertices. This means that $2e \leq Mv$ or that $2e/v \leq M$.

(c) Suppose G has n vertices. Since G has 15 edges and $\overline{G}$ has 13 edges, we know the complete graph on n vertices must have $15 + 13 = 28$ edges. Now the complete graph on n vertices has $n(n-1)/2$ edges. (One way to see this is to observe that vertex 1 needs to be connected to $2, 3, \ldots, n$ (total of $n-1$ edges), vertex 2 needs to be connected to $3, 4, \ldots, n$ (total of $n-2$ edges) and so on, giving a total of $(n-1) + (n-2) + \cdots + 1 = n(n-1)/2$.) This means that we need to solve the equation $n(n-1)/2 = 28$, which can be easily seen to be $n = 8$.

Problem 3. **(a)** We will list the tasks for each time slot (since the number of processors is unlimited).

Time slot 1: Foundation. (1 processor)
Time slot 2: Framing. (1 processor)
Time slot 3: Roof. (1 processor)
Time slot 4: Exterior siding (1 processor)
Time slot 5: Plumbing, Wiring, Exterior painting (3 processors)
Time slot 6: Flooring, Wallboard, Exterior fixtures (3 processors)
Time slot 7: Carpeting, Interior painting (2 processors)
Time slot 8: Interior fixtures (1 processor)
Time slot 9: Completion

(b) Without any other information, the bound is $2t$, This is because each task now takes time at most 2. So if we run the process in such a way that a processor waits (rests) for one unit of time when it runs a task that takes only one time unit, we have effectively put ourselves in a situation where each task takes 2 units of time.

We can exploit the information more optimally by doing the following: Replace each task that takes time 2 with two tasks that each take time 1, and where the first half needs to be completed before the commencement of the second half. This then gets us into a situation where we can use the theorem directly to actually obtain a way to schedule the tasks in stricly less than time $2t$ if such a way exists.

Problem 4. **(a)** Let $G = (V, E)$ be an undirected graph, where $|V| = n$. Assume G is represented by an adjacency matrix, M. Then, for every pair of vertices, (u, v), the number of even length paths of length $\leq n$, can be obtained from $\sum_{i=1}^{\lfloor n/2 \rfloor} M^{2i}$, where the (u, v) entry of every such matrix, M^{2k}, $1 \leq k \leq \lfloor n/2 \rfloor$, is the number of distinct paths of length exactly $2k$ from u to v in G (excluding cycles). For proof of correctness, see Theorem 2 on page 468.

(b) Let *EVEN* be the algorithm in part (a) except it now returns TRUE whenever there exists an even-length path between a pair of vertices (u, v). Let *ODD* be an algorithm that returns FALSE whenever there exists an odd-length path between a pair of vertices (u, v), by checking $\sum_{i=1}^{\lfloor n/2 \rfloor} M^{2i-1}$. Note that with *EVEN* and *ODD*, as soon as an even-length and an odd-length path is found, we return the logical value TRUE and FALSE respectively. Then we have:
$P(u, v) = \text{TRUE} \iff \mathit{EVEN} \ \wedge \ \neg \mathit{ODD} = \text{TRUE}$.

(c) If the graph $G = (V, E)$ is bipartite, then there are two sets V_1, V_2 of vertices such that no two vertices in V_1 and no two vertices in V_2 are adjacent. For every pair of vertices, (u, v), if $u \in V_1$ and $v \in V_2$ (or vice versa), and they share an edge, then there can never be an even-length path between them, and hence $P(u, v)$ is always false. If u and v are both in V_1 or both in V_2, then there can never be an odd-length path between them, and hence the problem reduces to finding an even-length path as we did in part (a).

Problem 5. Vertices c and d are the cut vertices. The removal of either one creates a graph with two components. The removal of any other vertex does not disconnect the graph.

Problem 6. Under these conditions, the matrix has a block structure, with all the 1's confined to small squares (of various sizes) along the main diagonal. The reason for this is that there are no edges between different components. The only 1's occur inside the small submatrices (but not all the entries in these square are 1's, of course.)

Problem 7. **(a)** For two nodes to have the same color, they must belong to the same MIS, since we color all the vertices of an MIS with the same color, and discard the used color so we never use that color again. But by definition, for any two vertices to be in the same MIS, they must not be incident on the same edge in the graph. Hence, with this algorithm, we never color the two endpoints of an edge with the same color.

(b) *Proof.* By induction on d: Let $P(d)$ denote the statement that any undirected graph with degree $\leq d$ is $d + 1$-colorable.
Base Case: $P(0)$ is clearly true since any graph with 0 degree is 1-colorable. $P(1)$ is also true since any graph with degree ≤ 1 is 2-colorable.
Inductive Case: We show for all d, $P(d) \Rightarrow P(d + 1)$.
Let G be any undirected graph with degree $\leq d + 1$. Choose any MIS, V' in G (note that an MIS always exists: if the graph is complete, an MIS consists of only one node in the graph). Color all the vertices in V' using the same color. Now reduce the graph by removing all the vertices in V' and their incident edges, and discard the color. The degree of all the vertices in $V - V'$ must be reduced by ≥ 1, since every vertex $\notin V'$ is adjacent to some vertex $\in V'$. So the reduced graph has degree $\leq d$, and thus by the induction hypothesis is $d + 1$-colorable. Adding the color that was used to color V' to the set of $\leq d + 1$ colors, implies that graph G with degree $d + 1$ is $d + 2$-colorable. ∎

Problem 8. We let the vertices of a graph be the animals, and we draw an edge between two vertices if the animals they represent cannot be in the same habitat because of their eating habits. A coloring of this graph gives an assignment of habitats, where the colors are the habitats.

Problem 9. Such a tree does exist. By Theorem $4(iii)$, we note that such a tree must have $i = 75/(m-1)$ internal nodes. This has to be a whole number, so $m-1$ must divide 75. This is possible, for example, if $m = 6$, so let us try it. A complete 6-ary tree of height 2 would have 36 leaves. We therefore need to add 40 leaves. This can be accomplished by changing 8 vertices at level 2 to internal vertices; each such change adds 5 leaves to the tree (6 new leaves at level 3, less the one leaf at level 5 that has been changed to an internal vertex). We will not show a picture of this tree, but just summarize its appearance. The root of the remaining 8 vertices at level 2 has 6 children, giving 36 vertices at level 2. Of course, 28 are leaves, and each of the remaining 8 vertices at level 2 has 6 children, living at level 3, for a total of 48 leaves at level 3. The total number of leaves is therefore $28+48 = 76$, as desired.

Problem 10. **(a)** *Proof.* By contradiction:
Suppose that not all the nodes in the graph have the same value. Then by the well-ordering property there is a smallest value, s, among the values of the nodes. Note that this value may not be unique in the graph. Consider a node v with value s. This node is connected to another node with value $> s$. This is true because s is the smallest value in the graph, and G is connected. But this implies that the average of the values of neighbors of v must be $> s$. But s is the average of all such values. Contradiction! Hence, in a connected graph if the value of every node is the *average* of its neighbors values, then all nodes must have the same value. ∎

(b) No the Theorem is no longer valid if the value of every node is the *median* of the values of its neighbors. We give the following counterexample.