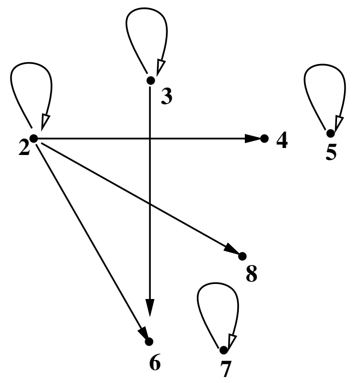
Massachusetts Institute of Technology — Handout 14
6.042J/18.062J: Mathematics for Computer Science — October 8, 1996
Professors Nancy Lynch and Tom Leighton

Problem Set 4 Solutions

Problem 1. **(a)** The set of ordered pairs $= \{(2,2),(2,4),(2,6),(2,8),(3,3),(3,6),(5,5),(7,7)\}$.

$$M_R = \left| \begin{array}{ccccccc} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right|$$

G =



(b) R is not reflexive or symmetric. It is antisymmetric and transitive. Hence, R is not an equivalence relation.

Problem 2. **(a)**

- $f(x) = e^x$ is an injection. It is not a surjection since $f : \mathbb{R} \longrightarrow \mathbb{R}^+$. Hence, it is not a bijection, and so it does not have an inverse.
- $f(x) = x^2 + 3$ is neither an injection, nor a surjection. Hence, it is not a bijection, and so it does not have an inverse.
- $f(x) = \frac{x^2+1}{x^2+2}$ is not an injection, surjection, or bijection. Hence, it does not have an inverse.

(b) Let $f : B \longrightarrow C$ and $g : A \longrightarrow B$, so that $f \circ g : A \longrightarrow C$, for sets $A, B,$ and C. Let x and y be distinct elements of A. Since g is one-to-one, $g(x)$ and $g(y)$ are distinct elements of B. Since f is one-to-one, $f(g(x)) = f \circ g(x)$ and $f(g(y)) = f \circ g(y)$ are distinct elements of C. Hence, $f \circ g$ is one-to-one.

Problem 3. Let $f : B \longrightarrow C$ and $g : A \longrightarrow B$, so that $f \circ g : A \longrightarrow C$, for sets $A, B,$ and C. Then:

(a) **Theorem 0.1** If $f \circ g$ is one-to-one, then g is one-to-one.

Proof. (By contradiction): Suppose not. Then there are distinct elements $a_1, a_2 \in A$ such that $g(a_1) = g(a_2)$. But this implies that $f(g(a_1)) = f(g(a_2)) \longrightarrow (f \circ g)(a_1) = (f \circ g)(a_2)$. Contradiction! $f \circ g$ is one-to-one by assumption. ■

(b) No, if $f \circ g$ is one-to-one, then f is not necessarily one-to-one. Consider the following counterexample.
Let $A = \{\Box, \Diamond, \clubsuit\}$, $B = \{a, b, c, d\}$, and $C = \{1, 2, 3\}$. Now suppose $g(\Box) = a$, $g(\Diamond) = b$, and $g(\clubsuit) = c$. Also, define $f(a) = 1$, $f(b) = 2$, $f(c) = f(d) = 3$. Then $f \circ g$ is clearly one-to-one, since $f \circ g(\Box) = 1$, $f \circ g(\Diamond) = 2$, and $f \circ g(\clubsuit) = 3$, but f is not one-to-one.

(c) This follows immediately from the definition. We want to show that $((f \circ g) \circ (g^{-1} \circ f^{-1})(z) = z,\ \forall z \in \mathbb{Z}$ and that $((g^{-1} \circ f^{-1}) \circ (f \circ g))(x) = x,\ \forall x \in X$. For the first we have:

$$\begin{aligned}
((f \circ g) \circ (g^{-1} \circ f^{-1}))(z) &= (f \circ g)((g^{-1} \circ f^{-1})(z)) \\
&= (f \circ g)(g^{-1}(f^{-1}(z))) \\
&= f(g(g^{-1}(f^{-1}(z)))) \\
&= f(f^{-1}(z)) \\
&= z
\end{aligned}$$

The second equality is done in a similar manner.

(d) $\sqrt[1]{\sqrt[2]{\ldots \sqrt[n-1]{\sqrt[n]{x}}}} = \sqrt[n!]{x}$

Problem 4. *Proof.* We have a listing of each of the sets S_i. Let s_{ik} be the kth element in the listing of S_i. We combine them, using the idea of dove-tailed interleaving shown in class, to get the listing for their union. The resulting list is $\{s_{11}, s_{12}, s_{21}, s_{13}, s_{22}, s_{31}, \ldots\}$. It is easy to see s_{ik} will occur before position $(i + k)^2$ in our listing. Of course, we also drop repeat elements from our listing but that does not affect anything. Since we have been able to give a valid listing, $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is countable. ■

Problem 5. **(a)** Uncountable.

Proof. (By contradiction): Suppose that the set of irrational but real numbers is countable. We know the set of real numbers, $\mathbb{R}$ is the union of the two sets of rational and irrational numbers. We also know that the set of rational numbers, $\mathbb{Q}$ is countable. Hence, it must be that $\mathbb{R}$ is countable, since the union of two countable sets is countable. Contradiction! So, the set of irrational but real numbers is uncountable. ■

(b) Countable. We showed in class that the set of rational numbers, $\mathbb{Q}$ is countable. Similarly, we can show the set of numbers of the form $b\sqrt{2}$, b rational, is countable by establishing a one-to-one correspondence between $\mathbb{Q}$ and the set of numbers of the form $b\sqrt{2}$: $\forall\ r \in \mathbb{Q}$, correspond r to $r\sqrt{2}$. Now, we show the set of numbers of the form $a + b\sqrt{2}$ is also countable. List the rational numbers vertically, and the numbers of the form $b\sqrt{2}$ horizontally to get a listing of all numbers of the form $a + b\sqrt{2}$ as follows:

	$\sqrt{2}$	$\sqrt{2}/2$	$2\sqrt{2}$	$3\sqrt{2}$	$\ldots$
1	$1+\sqrt{2}$	$1+\sqrt{2}/2$	$1+2\sqrt{2}$	$1+3\sqrt{2}$	$\ldots$
$1/2$	$1/2+\sqrt{2}$	$1+\sqrt{2}/2$	$1/2+2\sqrt{2}$	$1/2+3\sqrt{2}$	$\ldots$
$2/1$	$2+\sqrt{2}$	$2+\sqrt{2}/2$	$2+2\sqrt{2}$	$2+3\sqrt{2}$	$\ldots$
$3/1$	$3+\sqrt{2}$	$3+\sqrt{2}/2$	$3+2\sqrt{2}$	$3+3\sqrt{2}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The resulting set, obtained by dovetailing through, is $\{1 + \sqrt{2}, 1/2 + \sqrt{2}, 1 + \sqrt{2}/2, \ldots\}$, which is clearly countable.

(c) Uncountable. (See section 1.7, example 12, pages 77 – 78.)

Problem 6. **(a)** For solution, see section 4.2, example 8, page 246 (just replace n by 100).

(b) The set $\mathcal{P}$ must contain one of the n pairs of consecutive integers: $1, 2$ or $3, 4$ or $5, 6$ or $\ldots$ or $2n - 1, 2n$, which are clearly relatively prime. Let the set of $n + 1$ integers represent the $n + 1$ pigeons, and n pairs of consective integers that are relatively prime, represent the holes. Then by pigeonhole principle, there must exist at least one pair of consecutive integers among $n + 1$ integers in $\mathcal{P}$.

(c) First look at the k sums $a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + \ldots + a_k$. If one of these is divisible by k, we need not look any further. If not, each of these has a nonzero remainder when divided by k. Since the only possible nonzero remainders are $1, 2, 3, \ldots, k - 1$, by pigeonhole principle two of these sums must have the same remainder r when divided by k. The difference of two such integers is: $(qk+r)-(pk+r) = (q-p)k$, for integers p, q, which is divisible by k. In that case, if $a_1 + a_2 + \ldots + a_n$ and $a_1 + a_2 + \ldots + a_m$ are the sums in the question and $n > m$, then $a_{m+1} + a_{m+2} + \ldots + a_n$ is the required sum that is divisible by k.

Problem 7. *Proof.* (By constructing an injection from $[0, 1)$ to the function space):
We proved in problem 5, part (c), that the set of real numbers in $(0, 1)$ is uncountable. Let

us associate to each real number in this range (including 0 but not 1) a function from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ as follows: If x is a real number whose decimal representation is $0.d_1d_2\ldots$ (with ambiguity resolved by forbidding the decimal to end with an infinite string of $9'$s), then we associate to x the function whose rule is given by $f(n) = d_n$, $n \in \mathbb{Z}^+$. Clearly this is a one-to-one function from the set of real numbers in $(0, 1)$, and a subset of the set of all functions from the set of positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Two different real numbers must have different decimal representations, so the corresponding functions are different. (A few functions are left out, because of forbidding representations such as $0.239999\ldots$). Since the set $(0, 1)$ is uncountable, the subset of functions we have associated wiht them must be uncountable. But the set of all such functions has at least this cardinality, so it, too, must be uncountable. ■

Some common mistakes in your solutions

Problem 1(b) Some missed the fact that R is transitive.

Problem 2: Many thought that if a function is injective, then it must have an inverse. Hence, they concluded that $\ln x$ is the inverse of e^x.

Problem 3(a): Many students were confused about this definitional problem. Some tried to prove both f and g were injective, and some tried to show neither were injective.

Problem 5: Some did not provide a way to list the union of sets, they just said that the union of countable sets is countable but did not consider that the problem is about the union of *countably many* countable sets.

Problem 6(c): Some made their own assumptions about the numbers in the sequence (*e.g.*, they are consective integers, they are all distinct, etc). Some presented very lengthy and confusing solutions.