Practice Quiz 1

(Quiz #1 and Solutions, Fall 1995)

Problem 1. **(a)** Use Euclid's algorithm to find the *gcd* of 241 and 178.

	241	178	65	−88
1	178	63	−25	65
2	63	52	19	−23
1	52	11	−4	19
4	11	8	3	−4
1	8	3	−1	3
2	3	2	1	−1
1	2	1	0	1
2	1	0	1	0

Hence, $gcd(241, 178) = 1$.

(b) Find an integer solution to $241x + 178y = 10$.

From part a) we have

$$gcd(241, 178) = 65 \cdot 241 - 88 \cdot 178 = 1.$$

Multiplying by 10 we get

$$650 \cdot 241 - 880 \cdot 178 = 10.$$

Hence, one solution is $x = 650$ and $y = -880$.

Problem 2. Let S be a set of n positive integers. For any subset $A \subset S$ let $\Sigma(A)$ denote the sum of the elements in A. You are given that $\Sigma(S) \leq 2^n - 2$.

(a) Prove that there exist two distinct non-empty sets A and B such that $\Sigma(A) = \Sigma(B)$.

Proof. Consider $\mathcal{C} = \mathcal{P}(S) - \{S, \phi\}$. This is the collection of all nonempty subsets of S, excluding S. $|\mathcal{C}| = 2^n - 2$. For any $A \in \mathcal{C}$ it is the case that $1 \leq \Sigma(A) \leq 2^n - 3$. It is clear that $1 \leq \Sigma(A)$ since S is a set of positive integers and A is a nonempty subset of it. It is also clear that $\Sigma(A) \leq 2^n - 3$, since A is a proper subset of S and hence $\Sigma(A) < \Sigma(A) \leq 2^n - 2$.

Thus we have $2^n - 2$ elements A in $\mathcal{C}$ such that $1 \leq \Sigma(A) \leq 2^n - 3$. By the pigeonhole principle, we have that there exist two sets $A, B \in \mathcal{C}$ such that $\Sigma(A) = \Sigma(B)$. By construction we have that A and B are distinct non-empty subsets of S. ∎

(b) Prove that there exist two distinct non-empty sets A and B such that $\Sigma(A) = \Sigma(B)$ and $A \cap B = \phi$.

Proof. Let A' and B' be the two sets from part a). It is the case that $A' \not\subset B'$ and $B' \not\subset A'$ because for any two sets $P, Q \subset S$ it is the case that $P \subset Q \Rightarrow \Sigma(P) < \Sigma(Q)$.

Now consider, $A = A' - A' \cap B'$ and $B = B' - A' \cap B'$. It is clear that $\Sigma(A) = \Sigma(B)$ and $A \cap B = \phi$. Hence we are done. ■

Problem 3. [20 points] Let $(a, b) \sim (c, d)$ if $a + d = b + c$.

(a) [10 points] Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

- *Reflexivity* $(a, b) \sim (a, b)$ since $a + b = b + a$.
- *Symmetry* $(a, b) \sim (c, d) \Rightarrow (c, d) \sim (a, b)$ since $a + d = b + c \Rightarrow c + b = d + a$.
- *Transitivity* Assume $(a, b) \sim (c, d)$, i.e. $a + d = b + c$. And, $(c, d) \sim (e, f)$, i.e. $c + f = d + e$. Adding we get $a + d + c + f = b + c + d + e$ or $a + f = b + e$, i.e. $(a, b) \sim (e, f)$. Hence $(a, b) \sim (c, d) \wedge (c, d) \sim (e, f) \Rightarrow (a, b) \sim (e, f)$.

Since $\sim$ is reflexive, symmetric and transitive it is an equivalence relation.

(b) Partition the set $\{0, 1, 2\} \times \{0, 1, 2\}$ into the distinct equivalence classes under $\sim$. We can rewrite $(a, b) \sim (c, d)$ if $a + d = b + c$ as $(a, b) \sim (c, d)$ if $a - b = c - d$.

This shows that with any equivalence class $\mathcal{C}$ we can associate a *difference* d such that $(a, b) \in \mathcal{C} \Rightarrow a - b = d$. This gives us an easy way to list the equivalence classes of $\{0, 1, 2\} \times \{0, 1, 2\}$.

- $d = -2 : \{(0, 2)\}$.
- $d = -1 : \{(0, 1), (1, 2)\}$.
- $d = 0 : \{(0, 0), (1, 1), (2, 2)\}$.
- $d = 1 : \{(1, 0), (2, 1)\}$.
- $d = 2 : \{(2, 0)\}$.

(c) How many distinct equivalence classes under $\sim$ are there on the set $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$. You do not need to prove your answer.

$2n + 1$ since the difference can be any value in $\{-n, \ldots, n\}$.

Problem 4. Prove by induction that

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i \right)^2.$$

Let $P(n)$ be "$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$."

Base case. $P(1)$ is true since $\sum_{i=1}^{1} i^3 = 1 = \left(\sum_{i=1}^{1} i\right)^2$.

Inductive step. Assume $P(n)$ is true. Then

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2.$$

Adding $(n+1)^3$ to both sides we get,

$$\begin{aligned}
\sum_{i=1}^{n+1} i^3 &= \left(\sum_{i=1}^{n} i\right)^2 + (n+1)^3 \\
&= (n(n+1)/2)^2 + (n+1)^3 \qquad \text{since } \sum_{i=1}^{n} i = n(n+1)/2 \\
&= (n+1)^2/4 \times (n^2 + 4(n+1)) \\
&= ((n+1)(n+2)/2)^2 \\
&= \left(\sum_{i=1}^{n+1} i\right)^2
\end{aligned}$$

Hence, $P(n+1)$ is true. Thus $P(n)$ is true for all n.

Problem 5. n teams play a tournament. Every team plays every other team exactly once and all games end in a win/loss. It is possible that weaker teams can beat stronger teams.

(a) Any tournament can be viewed as a relation on the set of teams, with (A, B) in the relation iff A beats B. Is this relation a partial order for all tournaments? If yes, prove it. If not list all partial order properties that can fail.

Reflexivity fails since (A, A) does not belong to the relation of the tournament. Transitivity can fail since it could be in a tournament that A beats B who beats C who in turn beats A, i.e. (A, B) and (B, C) could be in the relation of the tournament without there also being (A, C).

(Note that anti-symmetry holds vacuously, since we never have both (A, B) and (B, A) in the relation.)

(b) Show (by induction) that in any tournament there exists a two-step champion, i.e. there exists a team A such that for every other team B

$$\text{it is the case that } \begin{cases} A \text{ has beaten } B, \text{ or} \\ \text{there exists } C \text{ such that } A \text{ has beaten } C \text{ who has beaten } B. \end{cases}$$

Proof. Let $P(n)$ be "in any tournament with n teams there exists a two-step champion."

Base case. $P(2)$ is true since in a tournament with only two teams the winner is a two-step champion.

Inductive step. Assume $P(n)$ is true. Consider a tournament with $n+1$ teams numbered $1, 2, \ldots, n+1$. Then the games between teams $1, 2, \ldots, n$ forms a subtournament and hence, by the inductive hypothesis, it must be the case that this subtournament possesses a two-step champion. Let this two-step champion be the team numbered 1 without loss of generality. Now, if $n+1$ loses to 1 or any of the teams that lost to 1 then it is clear that 1 is a two-step champion for the whole tournament. Otherwise, it is the case that $n+1$ beat 1 and all the teams that lost to 1 which makes $n+1$ the two-step champion. In either case the tournament possesses a two-step champion. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all n. ■

Problem 6. **(a)** Show that the set of real numbers $\in (0, 1)$ that have only 3s and 7s in their decimal representation is uncountable.

Proof. Suppose the set $\mathcal{C}$ of real numbers $\in (0, 1)$ that have only 3s and 7s in their decimal representation is countable. Then there exists a bijection $f : \mathbb{N} \to \mathcal{C}$. Represent this by a matrix M where $M_{ij} = 3$ if the jth digit of $f(i)$ (after the decimal point) is 3, and $M_{ij} = 7$ otherwise. Now create a new number T by considering the diagonal elements M_{ii} of the matrix. ith digit of $T = 3$ if $M_{ii} = 7$, otherwise ith digit of $T = 7$. This is a valid element of $\mathcal{C}$ and $f^{-1}(T)$ must exist. But observe that if the $f^{-1}(T)$th digit of $f^{-1}(T)$ is 3 then the $f^{-1}(T)$th digit of $f^{-1}(T)$ must be 7 and vice-versa. Thus, either way, we have a contradiction. Hence $\mathcal{C}$ cannot be countable. ■

(b) Prove that the cross product of two countable sets is countable.

Proof. Let the two sets be $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$. We can list the elements of the sets in this fashion because they are countable and hence have a bijection with the naturals. We list the cross product, using the idea of dovetailing shown in class, to get the listing $\{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), \ldots\}$. It is easy to see that (a_i, b_j) will occur before position $(i+j)^2$ in our listing. Since we have been able to give a valid listing $A \times B$ is countable. ■