Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 10
October 1, 1996

Problem Set 5

Due: Start of Class, Thursday, October 10.

Reading: Sections 6.6, 7.2–7.5, 7.8 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 1. **[8 points]** (Problem 22, Sec. 6.6 of Rosen)

Answer the following questions concerning the poset $(\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}, |)$.

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{2, 9\}$.

(f) Find the least upper bound of $\{2, 9\}$, if it exists.

(g) Find all lower bounds of $\{60, 72\}$.

(h) Find the greatest lower bound of $\{60, 72\}$, if it exists.

Problem 2. **[9 points]** (Problems 26 and 34, Sec. 7.2 of Rosen)

Let G be a graph with v vertices and e edges. Let M be the maximum degree of the vertices fo G, and let m be the minimum degree of the vertices of G. Show that

(a) $2e/v \geq m$.

(b) $2e/v \leq M$.

(c) If G is a simple graph with 15 edges and $\overline{G}$ has 13 edges, how many vertices does G have?

Problem 3. **[10 points]**

Recall the theorem given in class that given a task graph that is a partial order with maximum chain length t, we can perform the entire set of tasks, using an unlimited collection of parallel processors in time t.

(a) Apply this theorem to produce an optimal schedule for the task graph on page 419 of Rosen.

(b) Give an upper bound on the parallel time for a generalization of this theorem, where some tasks take 1 unit of time, and others take 2 units of time.

Problem 4. **[18 points]**

(a) Give an algorithm to determine, given an undirected graph G, for all pairs of vertices, (u, v), how many even-length paths there are from u to v. Explain why your algorithm is correct. Assume the graph is represented by an adjacency matrix.

(b) Give an algorithm to determine, given an undirected graph G, for all pairs of vertices, (u, v), whether or not the following predicate $P(u, v)$ is true:

$P(u, v)$: There is an even-length path, but no odd-length path, from u to v. Explain why your algorithm is correct.

(c) Repeat (b), but now assume that your graph is bipartite. Can you design a more efficent algorithm than your general algorithm in part (b)?

Problem 5. **[6 points]**

Do Exercise 7.4 – 16 on page 470 of Rosen.

Problem 6. **[10 points]**

Do Exercise 7.4 – 30 on page 472 of Rosen.

Problem 7. **[16 points]**

An *independent set* of a graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices such that each edge in E is incident on at most one vertex in V'. A *maximal independent set*, or *MIS*, is an independent set V' such that for all vertices $v \in V - V'$, the set $V' \cup \{v\}$ is not independent, *i.e.*, every vertex not in V' is adjacent to some vertex in V'.

Consider an alternative graph coloring algorithm to the one given in class, as follows:

Given an undirected graph G with degree $\leq d$, choose any *MIS*, V' of G. Color all the vertices of V' using the same color. Now discard the used color, and reduce the graph G by removing all the colored vertices and their incident edges. Color the vertices of the remaining graph with the remaining colors using the same procedure in a recursive manner.

(a) Show that with this algorithm, we never color the two endpoints of an edge with the same color.

(b) Prove by induction on d, that the algorithm uses $\leq d+1$ colors to color any undircted graph with degree $\leq d$.

Problem 8. **[7 points]** (Problem 18, Sec. 7.8 of Rosen)

A zoo wants to set up natural habitats in which to exhibit its animals. Unfortunately, some animals will eat some of the others when given the opportunity. How can a graph model and a coloring be used to determine the number of different habitats needed and the placement of the animals in these habitats?

Problem 9. **[6 points]**

Do Exercise 8.1 – 18 on page 546 of Rosen.

Problem 10. **[10 points]**

(a) Prove the following theorem:
Theorem: Given a connected graph $G = (V, E)$, where every node in the graph is associated with a positive integer, *i.e.*, there is a function $f : V \longrightarrow \mathbb{Z}^+$, prove that if the value of every node is the "*average*" (mean) of its neighbors values, then all nodes must have the same value.

(b) Is the theorem still valid if the value of every node is the "*median*" of its neighbors values? Either prove it, or give a counterexample. (Definition: A median of a set of even size is either of the two middle numbers. For example, if $A = \{1, 2, 3, 4\}$, then either 2 or 3 is a median of the set A.)