Problem Set 4

Due: Start of Class, Tuesday, October 1.

Reading: Sections 1.6, 1.7 (pages 76-78), 4.2, 6.2, and 6.3 of *Discrete Mathematics and its Applications* by Kenneth H. Rosen.

Problem 1. [14 points]

Let R be the relation $\{(a,b)|a$ is prime, and $a|b\}$ on the set $A = \{2,3,4,5,6,7,8\}$.

(a) [**7 points**] List the ordered pairs, write the matrix, and draw the directed graph representing R.

(b) [**7 points**] Determine whether R is reflexive, symmetric, antisymmetric, and transitive. Is R an equivalence relation?

Problem 2. [10 points]

(a) [**6 points**] Determine whether each of the following functions, $f : \mathbb{R} \to \mathbb{R}$, is an injection, surjection or a bijection. Also, give the inverse if it exists.

- $f(x) = e^x$
- $f(x) = x^2 + 3$
- $f(x) = (x^2+1)/(x^2+2)$

(b) [**4 points**] Prove that the composition of two injective functions is injective.

Problem 3. [15 points]

(a) [**5 points**] If $f \circ g$ is one-to-one, does it follow that f and g are also one-to-one? Justify your answer.

(b) [**5 points**] Suppose that f is an invertible function from Y to Z and g is an invertible function from X to Y. Show that the inverse of the composition $f \circ g$ is given by $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. (Problem 36, Sec. 1.6 of Rosen)

(c) [**5 points**] Let $f_1, f_2, ..., f_n$ be functions from $\mathbb{N}$ to $\mathbb{N}$, where $f_i(x) = \sqrt[i]{x}$.
Compute $f_1 \circ f_2 \circ ... \circ f_n$.

Problem 4. **[15 points]** Determine whether each of the following sets is countable or uncoutable. Prove your answers.

(a) **[5 points]** The set of irrational but real numbers.

(b) **[5 points]** The set of numbers of the form: $a + b\sqrt{2}$, where a and b are rational.

(c) **[5 points]** The set of real numbers in $(0, 1)$.

Problem 5. **[10 points]** Let $\mathcal{S}_i, 1 \leq i \leq \infty$ be a family of countable sets. Prove that $\bigcup_{i=1}^{\infty} \mathcal{S}_i$ is countable.

Problem 6. **[21 points]**

(a) **[7 points]** Show that any subset of 101 numbers from [1,200] contains two numbers x and y such that $x|y$.

(b) **[7 points]** If $\mathcal{P}$ is any set of $n + 1$ integers selected from $1, 2, 3, \ldots, 2n$, prove that there are two relatively prime integers in $\mathcal{P}$.

(c) **[7 points]** Consider a sequence of k integers: $a_1, a_2, \ldots, a_k$. Show that there is some subset of these terms whose sum is divisible by k.

Problem 7. **[15 points]**

Use a diagonalization argument to show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable. (Problem 30, Sec. 1.7 of Rosen)