Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
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Handout 8
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Problem Set 2 Solutions

Problem 1. **Base Case:** $n = 1$. True since $1 \cdot 1! = 2! - 1$.
Inductive Step: Assuming the inductive hypothesis, we have

$$\begin{aligned} 1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! + (n+1) \cdot (n+1)! &= (n+1)! - 1 + (n+1) \cdot (n+1)! \\ &= (n+1)!(1 + n + 1) - 1 \\ &= (n+2)! - 1 \end{aligned}$$

as desired.

Problem 2. **Base Case:** $n = 1$. True since $f_1^2 = f_1 f_2 = 1$.
Inductive Step: Assume it is true for $k \leq n$. Then we have

$$\begin{aligned} f_1^2 + f_2^2 + ... + f_n^2 + f_{n+1}^2 &= f_n f_{n+1} + f_{n+1}^2 \\ &= f_{n+1}(f_n + f_{n+1}) \\ &= f_{n+1} f_{n+2} \end{aligned}$$

which is true by the inductive hypothesis and definition of *Fibonacci* numbers.

Problem 3. The max or min of one number is itself. $\max(a_1, a_2) = a_1 \bigvee a_2$ if $a_1 \geq a_2$, and $a_1 < a_2$ respectively. Also, $\min(a_1, a_2) = a_1 \bigvee a_2$ if $a_2 \geq a_1$ and $a_2 < a_1$ respectively. For $n \geq 2$ we define

$$max(a_1, a_2, ..., a_{n+1}) = max(max(a_1, a_2, ..., a_n), a_{n+1})$$

and

$$min(a_1, a_2, ..., a_{n+1}) = min(min(a_1, a_2, ..., a_n), a_{n+1})$$

Problem 4. First, we prove the follwing theorem:

Theorem 0.1 Let n be a positive integer. Show that a $2^n \times 2^n \times 2^n$ cube, (an n-*cube*) with one arbitrary cubic foot removed can be built with basic blocks of size $2 \times 2 \times 2$ (1-*cube*) with one of its cubic foot missing.

Proof. Let $P(n)$ be the statement that any n-*cube* with one arbitrary cubic foot missing can be built using 1-*cubes* with one cubic foot removed. We use mathematical induction to prove $P(n)$ is true for all positive integers n.

Basis Step $P(1)$ is true since a 1-*cube* with one missing cubic foot is already provided as basic block by CIA.

Inductive Step: Assume that $P(n)$ is true. We show that $P(n+1)$ is true. Construct 8 n-*cubes* in such a way that 7 of them are missing one cubic foot from one of their 8 corners, and the 8*th* one is missing an arbitrary cubic foot. We can do this by the induction hypothesis. We construct an $(n+1)$-*cube* as follows: Put the 7 corner-missing n-*cubes* together so that all of their missing corners meet in the center. Then fill up the gap with a basic block. Now add the 8*th* n-*cube* with an arbitrary cubic foot missing to get an $(n+1)$-*cube* with an arbitrary cubic foot removed. ■

Now we are in good shape to prove the master theorem.

Theorem 0.2 Prove that an $(n+1)$-*cube* with one cubic foot missing from its center (the **CIA** cube) can be built using 1-*cubes* with one cubic foot missing.

Proof. The **CIA** cube can be built by following the steps in the proof of **Theorem** 0.1 except that when building the 8*th* cube, we choose the arbitrary missing cubic foot to be one of the 8 possible 1-*cubes* in the center of the 8*th* n-*cube*. ■

Problem 5. Let E denote the number of edges, and N be the number of nodes in a $3,4$-*tree*. Then:

(a) $E = N - 1$.

(b) **Theorem 0.3** In a $3,4$-*tree*, $E = N - 1$.

Proof. By Structural Induction.

Basis Step: Let $N = 1$. This it is clearly true since there are no edges in a one-node tree.

Inductive Step: Assume this is true for $k \leq n$. We need to show that it is also true for a tree with $N+1$ nodes. Let's break this into two cases.

Case 1: Root of the tree has 3 childeren, where every such subtree is a $3,4$-*tree* by definition. By the induction hypothesis, for every such subtree the number of edges is one less than the number of nodes in the subtree. Hence, for a tree with $N+1$ nodes, there are total of N nodes and $N-3$ edges in the 3 subtrees. The root-node adds one more node and 3 edges connecting to the 3 subtrees, making it a total of $N+1$ nodes and N edges as desired.

Case 2: Root of the tree has 4 childeren, where every such subtree is a $3,4$-*tree* by definition. By the induction hypothesis, for every such subtree the number of edges is one less than the number of nodes in the subtree. Hence, for a tree with $N+1$ nodes, there are total of N nodes and $N-4$ edges in the 4 subtrees. The root-node adds one more node and 4 edges connecting to the 4 subtrees, making it a total of $N+1$ nodes and N edges as desired. ■

Problem 6. **(a)** $Pink = \{x : x \in \mathbb{N},\ \text{and } x \geq 24\}$.

(b) **Theorem 0.4** Every natural number ≥ 24 can be expressed as the sum of, not necessarily distint, perfect squares of numbers ≥ 2.

Proof. By Contradiction: Define the set $Blue = \{x : x \in \mathbb{N}, x \geq 24, \text{ and } x \notin Pink\}$.

Then by the well-ordering property, $Blue$ has a smallest element, s. Now let $4pinks = \{24, 25, 26, 27\}$. Then $x \in 4pinks \longrightarrow x \in Pink$, since we can write:

$$\begin{aligned} 24 &= 6 \cdot 4 \\ 25 &= 4^2 + 9 \\ 26 &= 2 \cdot 4 + 2 \cdot 9 \\ 27 &= 3 \cdot 9 \end{aligned}$$

So, s must be ≥ 28. Since s is the smallest element in $Blue$, we have $s' = s - 4 \geq 24$, and $s' \in Pink$. But,

$$s' \in Pink \quad \longrightarrow \quad s' + 4 \in Pink \quad \longrightarrow s \in Pink$$

since $s' + 4$ is the sum of perfect squares ≥ 2. But it cannot be that $s \in Blue$, and $s \in Pink$, since $Blue \cap Pink = \emptyset$. Contradiction! Hence, $Blue$ is indeed empty, and so every natural number ≥ 24 is *pink*. ■

Problem 7. **(a)** We do not have enough base cases to support the claim. The proof breaks down when we assume $2^1 = 3^1$ by the induction hypothesis, which had to be shown to be true as a base case.

(b) The inductive hypothesis is involed wrongly in the inductive step. $n+1$ need not be composite; it could very well be a prime.

Some common mistakes in your solutions

Problems 1 and 2:

- some students gave extraneous base cases.
- a few forgot to argue the base case.

- a few refered to $P(n)$ without defining it.

Problem 3:

- several people forgot the $a_1 = a_2$ case.
- a few people forgot to define a base case.

Problem 4:

- almost everyone used circular reasoning.
- a few assumed that a $2^n \times 2^n \times 2^n$ block with one cubic foot missing could be built without offerring any proof of it.

Problem 5:

- several students just gave a few examples and didn't use induction.
- some people forgot the base case.
- several people identified steps that violated the $3, 4$-*tree* property as valid construction steps in their inductive argument.

Problem 6:

- a few people just gave a few examples but didn't prove it.
- a few displayed a fundamental lack of understanding of the well-ordering principle.

Problem 7:

- some people said "the math was wrong", *i.e.*, $2^1 \neq 3^1$, $\log_2 n \neq 0$ if $n \neq 0$, therefore the proof was false.
- several people showed that the theorem was false, not that the proof was invalid.