Problem Set 1 Solutions

Problem 1. **(a)** $p \Rightarrow r$.

(b) $\neg p \wedge q \wedge r$.

(c) $(p \wedge q) \Leftarrow r$.

(d) $(p \vee q) \Leftrightarrow r$.

Problem 2. $(p \vee \neg q \vee \neg r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$.

Problem 3. **(a)** $\exists x \exists y Q(x, y)$.

(b) $\neg \exists x \exists y Q(x, y)$ or $\forall x \forall y \neg Q(x, y)$ or $\forall x \neg \exists y Q(x, y)$

(c) $\forall y \exists x Q(x, y)$.

(d) $\exists x [Q(x, Jeopardy) \wedge Q(x, Wheel\ of\ Fortune)]$.

(e) $\exists x_1 \exists x_2 [Q(x_1, Jeopardy) \wedge Q(x_2, Jeopardy) \wedge x_1 \neq x_2]$.

Problem 4. **(a)** $\forall x \exists y\ F(x, y)$.

(b) $\neg \exists x \forall y\ F(x, y)$.

(c) $\forall y \exists x\ F(x, y)$.

(d) $\exists y_1 \exists y_2 [F(Nancy, y_1) \wedge F(Nancy, y_2) \wedge (y_1 \neq y_2) \wedge \forall y (F(Nancy, y) \Rightarrow (y = y_1 \vee y = y_2))]$.

(e) $\exists y [\forall x F(x, y) \wedge \forall z (\forall x F(x, z) \Rightarrow z = y)]$.

(f) $\neg \exists x F(x, x)$.

Problem 5. **(a)** $\forall x (P(x) \Rightarrow Q(x))$.

(b) $\exists x (R(x) \wedge \neg Q(x))$.

(c) $\exists x (R(x) \wedge \neg P(x))$.

(d) Yes. The unsatisfactory excuse guaranteed by part **(b)** cannot be a clear explanantion by part **(a)**.

Problem 6. As a counter example, let $P(x)$ denote the statement "x is an even number", and $Q(x)$ be the statement "x is an odd number". Let the universe of discourse be the set of positive integers. Then $\forall x P(x) \bigvee \forall x Q(x)$ is clearly false. But $\forall x (P(x) \bigvee Q(x))$ is true, since every positive integer is either odd or even.

Problem 7. **(a)** Fallacy of affirming the conclusion. Let $p : n > 1, q : n^2 > 1$. Then $((p \Rightarrow q) \bigwedge q) \Rightarrow p$.

(b) Fallacy of begging the question. **Proof:** Suppose $\log_2 3$ is rational. Then by definition of rational numbers we have: $\log_2 3 = a/b$, where a and b are integers, and $\gcd(a, b) = 1$. But then $3 = 2^{a/b} \Rightarrow 3^b = 2^a \Rightarrow$ an odd number $=$ an even number. Contradiction! Hence, $\log_2 3$ is irrational.

(c) Valid argument using modus tollens. Let $p : n > 2, q : n^2 > 4$. Then $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$.

(d) Fallacy of denying the hypothesis. As a counter example consider the case when $n = -4$.

Problem 8. Proof by contradiction. Suppose that a/b is a nonzero rational number and that x is an irrational number. Suppose also that xa/b is rational. If so, then multiply this rational number by the rational number b/a. We can do this because a/b nonzero implies that a is nonzero, and hence b/a is a nonzero rational number. Now $xa/b \cdot b/a$ is also rational (see exercise 12, sec 3.1). But $xa/b \cdot b/a = x$, which is irrational by hypothesis. This is a contradiction, so in fact xa/b must be irrational, as desired.

Problem 9. **Solution 1:** Suppose that $\sqrt{2} + \sqrt{3}$ is rational. Then it can be written as a/b, where a and b are integers, and $\gcd(a, b) = 1$. Moreover, $\sqrt{2} - \sqrt{3}$ is also rational since:

$$(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3}) = a(\sqrt{2} - \sqrt{3})/b$$

which implies

$$-1 = (\sqrt{2} - \sqrt{3})a/b \longrightarrow -b/a = \sqrt{2} - \sqrt{3}$$

and $-b/a$ is a nonzero rational number. Hence, it must be true that $(\sqrt{2}+\sqrt{3})+(\sqrt{2}-\sqrt{3})$ is rational, since the sum of two rational numbers is rational. But $(\sqrt{2}+\sqrt{3})+(\sqrt{2}-\sqrt{3}) = 2\sqrt{2}$, and $2\sqrt{2}$ is irrational by Problem 8. Contradiction! Therefore $\sqrt{2} + \sqrt{3}$ is irrational.

Solution 2: First we prove the following lemma.

Lemma 1: Prove that $\sqrt{6}$ is irrational.

Proof by contradiction: Suppose it is rational. Then $\sqrt{6} = a/b$, where a and b are integers, and $\gcd(a, b) = 1$. Then

$$6 = a^2/b^2 \longrightarrow 3(2b^2) = a^2$$

But a^2 is even, which implies a is even, which implies a^2 is divisible by 4. So write $a^2 = 4c^2$, which implies $3b^2 = 2c^2$. So b^2 is even, and is divisible by 4. So write $b = 2d$. Then we have $a/b = 2c/2d$. But $\gcd(a, b) = 1$. contradiction! Hence,$\sqrt{6}$ is irratinal.

Now assume that $\sqrt{2} + \sqrt{3}$ is rational. Then $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6} = a^2/b^2$, implies that $\sqrt{6} = (a^2/b^2 - 5)/2$, which makes $\sqrt{6}$ rational, since $(a^2/b^2 - 5)/2$ is rational. But by *Lemma 1*, $\sqrt{6}$ is irrational. Contradiction! Therefore, $\sqrt{2} + \sqrt{3}$ is irrational.

Problem 10. There are three cases to consider.

1. $x < y \Rightarrow max(x, y) = y$, and $x < y \Rightarrow min(x, y) = x$. Hence, $max(x, y) + min(x, y) = x + y$.
2. $x = y \Rightarrow max(x, y) = x \bigvee y$, and $x = y \Rightarrow min(x, y) = x \bigvee y$. Hence, $max(x, y) + min(x, y) = x + y$.
3. $x > y \Rightarrow max(x, y) = x$, and $x > y \Rightarrow min(x, y) = y$. Hence, $max(x, y) + min(x, y) = x + y = y + x$.

Therefore, $max(x, y) + min(x, y) = x + y$ for all real numbers x, y.

Some common mistakes in your solutions

Problem 3: Many people forgot to include $x_1 \neq x_2$ to signify the "at least".

Problem 4, parts (d) and (e): Almost everyone had trouble handling the "exactly".

Problems 8 and 9: Almost everyone assumed something that should have been proved, the most popular one being "$\sqrt{6}$ is irrational" in problem 9, when people tried to show that $(\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ is irratrional. A few people realized they needed to prove it, but didn't know how to.

Problem 10: Many students forgot that $x = y$ is also a case that had to be considered.

In general, there seems to be a lot of confusion about concepts such as universe of discourse, sets, and elements. Some people had statements such as $\forall x P(x), \exists a \in x$.