Problem Set 2

Due: Start of Class, Tuesday, September 17, 1996.

Reading: Sections 3.2-3.4. Background: 2.3

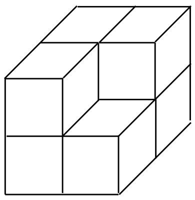
Problem 1. Prove that $1 \cdot 1! + 2 \cdot 2! + ... + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

Problem 2. Prove that $f_1^2 + f_2^2 + ... + f_n^2 = f_n f_{n+1}$, where f_n is the nth Fibonacci number and n is a positive integer.

Problem 3. Give a recursive definition of the functions max and min so that $max(a_1, a_2, ..., a_n)$ and $min(a_1, a_2, ..., a_n)$ are the maximum and minimum of the n numbers $a_1, a_2, ..., a_n$, respectively.

Problem 4. The C.I.A. wants you design plans to build (for an arbitrary $k \geq 1$) a $2^k \times 2^k \times 2^k$ cubic foot granite cube that is solid except for a missing single cubic foot that is one of the eight cubic feet nearest the center of the cube. It will have one of its corners exactly at the center of the large cube. The C.I.A. is not telling you what is to be placed inside the cube.

You are to build this by cementing together *basic blocks* that are 2-foot $\times$ 2-foot $\times$ 2-foot cubes, with one one-foot cube missing from its corner – see the diagram below.



Describe your method for doing this. Your plans should be described inductively on k.

Problem 5. In a $3,4$-*tree*, every node is either a *leaf node* with no children, or an *internal node* with exactly 3 or exactly 4 children.

(a) What is the relationship between the number of edges and the number of nodes in a $3, 4$-tree?

(b) Prove your conjecture from (a) using structural induction, that is, strong induction on the number n of recursive formation steps needed to construct the tree.

Problem 6. Define a natural number to be *pink* if it is:

1. Expressible as the sum of (not necessarily distinct) perfect squares of numbers ≥ 2,
2. and it is ≥ 24.

(a) Characterize exactly what the pink numbers are.
(b) Prove your answer to (a), using well ordering.

Problem 7. Find the fallacies in the proofs of the following theorems.

(a) *Theorem*: $2^n = 3^n$ for all natural numbern n.

BASIS STEP: $2^0 = 3^0 = 1$ by definition.

INDUCTIVE STEP: Assume that $2^k = 3^k$ for all natural numbers $k \leq n$. Then

$$2^{n+1} = 2^n * 2^1 = 3^n * 3^1 = 3^{n+1}$$

by inductive hypothesis.

(b) *Theorem*: $\log_2 n = 0$ for all natural numbers n.

BASIS STEP: $\log_2 1 = 0$ is true.

INDUCTIVE STEP: Assume that $\log_2 k = 0$ for all natural numbers $k \leq n$, and show it is true for $n + 1$. Write $n + 1$ as a product of two natural numbers p and q so that we have:

$$\log_2(n + 1) = \log_2(p.q) = \log_2 p + \log_2 q = 0 + 0 = 0$$

which is true by inductive hypothesis.