Massachusetts Institute of Technology | Handout 17
6.042J/18.062J: Mathematics for Computer Science | March 7, 1996
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Problem Set 4

Due: March 15, 1996.

Reading for this assignment: Rosen course text, §1.6, 1.7, 4.2, 7.1

Reading for recitation and next week's lectures: §6.3, 6.4, 7.1–5, 7.8

Problem 1. As on Quiz 1, let A be the set of nonempty subsets of $\{1, 2, \ldots, n\}$ for some integer $n > 1$. For $a \in A$, let $\max(a)$ be the maximum element in a. Define a partial order, $\sqsubseteq_A$, on A by the rule that

$$a \sqsubseteq_A a' \text{ iff } \quad \max(a) < \max(a') \text{ or } [\max(a) = \max(a') \text{ and } a \subseteq a'].$$

You learned from the quiz (we hope :-)) that the parallel-time of $(A \sqsubseteq_A)$, which equals the length of the longest chain, is $n(n+1)/2$. Now Dilworth's Theorem implies that there is an antichain of size at least $\lceil 2(2^n - 1)/n(n+1) \rceil$.

In fact there are much larger antichains when n is large enough. What is the least $n > 1$ such that there is an antichain of size strictly bigger than guaranteed by Dilworth's Theorem? Write down the elements of such an antichain.

Problem 2. Rosen, Problem 4.2.34, p. 249.

Problem 3. In class we have shown that for any set A, if there is an injection from A to a proper subset of A, then there is also an injection from $\mathbb{N}$ to A. Prove the converse: if there is an injection from $\mathbb{N}$ to A, then there is an injection from A to a proper subset of A.

Problem 4. Prove that the set $[0, 1)$ (the half-open unit interval of the real line) has the same cardinality as the set $[0, 1) \times [0, 1)$ (the half-open unit square of the Euclidian plane).

Hint: Use the fact that every real number $r \in [0, 1)$ has a *unique* decimal representation $0.d_1 d_2 d_3 \cdots$ which does not end in all 9's.

Problem 5. Let A be a set. Define the set $\text{seq}(A)$ to be the set of all finite tuples of one or more elements in A. Namely,

$$\text{seq}(A) = \bigcup_{n>0} \underbrace{A \times \cdots \times A}_{n}.$$

We say a set S is *countable* iff there is an injection from S to $\mathbb{N}$. Prove that seq($\mathbb{N}$) is countable.

Hint: Define the *norm* of any tuple $\langle n_1, \ldots, n_k \rangle \in \text{seq}(\mathbb{N})$ to be $\max\{k, n_1, \ldots, n_k\}$. Observe that for any $k > 0$, there are only finitely many tuples with norm k.

Problem 6. Define a *checkered* set of reals numbers to be one which is a finite union of disjoint, finite, closed intervals. Let "twotenths" be the operation which maps a checkered set to the checkered set obtained by removing all but the first and last tenth of each closed interval. For example,

$$\begin{aligned} \text{twotenths}([0,1]) &= [0,1/10] \cup [9/10,1], \\ \text{twotenths}([0,1/10] \cup [9/10,1]) &= [0,1/100] \cup [9/100,1/10] \cup [9/10,91/100] \cup [99/100,1]. \end{aligned}$$

Continuing in this way, we define

$$\begin{aligned} \mathfrak{C}_0 &= [0,1], \\ \mathfrak{C}_{n+1} &= \text{twotenths}(\mathfrak{C}_n), \\ \mathfrak{C} &= \bigcap_{n \geq 0} \mathfrak{C}_n. \end{aligned}$$

(a) Prove that for any $\epsilon > 0$, there is a checkered set of intervals of total length $< \epsilon$ which contains $\mathfrak{C}$.

(b) Prove that $\mathfrak{C}$ is not countable. (Hint: Consider the infinite decimal expansion of elements in $\mathfrak{C}$.)

Problem 7. **(a)** The *intersection graph* of a collection of sets $A_1, A_2, ..., A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these two sets have a nonempty intersection. Construct the intersection graph for the following collection of sets.

$$\begin{aligned} A_1 &= \{0,1\} \\ A_2 &= \{1,2\} \\ A_3 &= \{2,3,8\} \\ A_4 &= \{3,4,5\} \\ A_5 &= \{1,2,4,9\} \end{aligned}$$

(b) Show that a graph is *simple* (cf. Rosen, §7.1 ,Def. 1) iff it is graph isomorphic (cf. Rosen, §7.3, Def. 1) to an intersection graph.