Problem Set 3 Solutions

Problem 1. Rosen, 6.1.28 on page 366

(a) $R \bigcup S$ is reflexive.
Since R is reflexive, $(a, a) \in R$ for all $a \in R$. Thus, $(a, a) \in R \bigcup S$ for all $a \in R$. So, $R \bigcup S$ is reflexive.

(b) $R \bigcap S$ is reflexive.
Since R, S are reflexive, $(a, a) \in R$ and $(a, a) \in S$ for all $a \in R$. Therefore, $(a, a) \in R \bigcap S$. So, $R \bigcap S$ is reflexive.

(c) $R \oplus S$ is irreflexive.
(a, a) is both in R and S, so $(a, a) \notin R \oplus S$. Thus, $R \oplus S$ is irreflexive.

(d) $R - S$ is irreflexive.
(a, a) is both in R and S, so $(a, a) \notin R - S$. Thus, $R \oplus S$ is irreflexive.

(e) $S \circ R$ is reflexive.
$(a, a) \in R$ and $(a, a) \in S$, so $(a, a) \in S \circ R$. Thus, $S \circ R$ is reflexive.

Problem 2. Rosen, 6.5.26 on page 401

(a) This is a partition. Every number is either even or odd.

(b) Almost a partition. However, 0 is neither a positive or negative number, so 0 is not in either set. Thus, this is not a partition.

(c) This is a partition. Every number is congruent to either 0,1, or 2 (mod 3).

(d) This is a partition. If the absolute value of a number is less than or equal to 100, it's in the second set. Otherwise, it will be in either the first or third set.

(e) Not a partition. For example, 2 is not divisible by 3, and 2 is an even integer. Therefore, 2 is in two different sets.
These sets do cover all the integers. For an element, x, *not* to be in the first set, $x \equiv 0 (\text{mod } 3)$. For x *not* to be in the second set, $x \equiv 1 (\text{mod } 2)$. By the Chinese Remainder Theorem, this corresponds to the set $x \equiv 3 (\text{mod } 6)$, which is the third set.

Problem 3. Rosen, 6.6.20 on page 417

(a) Maximal elements – l,m.

(b) Minimal elements – a,b,c.

(c) No. There is no greatest element.

(d) No. There is no least element.

(e) Upperbounds for $\{a, b, c\}$ are k,l,m.

(f) The least upperbound for $\{a, b, c\}$ is k.

(g) There are no lower bounds for $\{f, g, h\}$.

(h) There is no greatest lowerbound.

Problem 4. (Refer to Handout 7.) Define the binary relation $=_\vdash$ on arithmetic expressions (ae's) by the rule:

$$e =_\vdash f \quad \text{iff} \quad \vdash (e = f).$$

Show that $=_\vdash$ is an equivalence relation on ae's.

Proof. To show $=_\vdash$ is an equivalence relation on ae's, we must show it is reflexive, symmetric, and transitive.

- **Reflexive** $\vdash e = e$ by the reflexivity rule. Therefore, $e =_\vdash e$.
- **Symmetric** If $e =_\vdash f$, then there is a proof that $e = f$. Adding the equation $f = e$ at the end of this proof is also a proof, since the last line now follows by the symmetry rule. So we have a proof that $f = e$. Hence, $f =_\vdash e$.
- **Transitive** If $e =_\vdash f$ and $f =_\vdash g$, then there is proof that $e = f$ and a proof that $f = g$. If put these two proofs together (i.e. concatenate the first proof to the front the second proof) followed by the equation $e = g$, we get a proof in which the last line follows by transitivity. So we have a proof of $e = g$. Hence, $e =_\vdash g$.

Therefore, $=_\vdash$ is an equivalence relation. ■

Problem 5. (A construction of the rational numbers.)

Let P be the set of all pairs of integers, where the second integer of the pair is nonzero, i.e. $P = \mathbb{Z} \times (\mathbb{Z} - \{0\})$. We define a relation $\sim$ on P as follows:

$$(a, b) \sim (c, d) \quad \text{iff} \quad ad = bc.$$

(a) Prove that $\sim$ is an equivalence relation.

We need to show $\sim$ is reflexive, symmetric, and transitive.

- **Reflexive** $(a, b) \sim (a, b)$, since $ab = ba$.
- **Symmetric** If $(a, b) \sim (c, d)$, then $ad = bc$. But then $cb = da$, so $(c, d) \sim (a, b)$.
- **Transitive** If $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $ad = bc$ and $cf = de$. So $(ad)f = (bc)f = b(cf) = b(de)$. Since $d \neq 0$, we cancel d and conclude $af = be$. Thus, $(a, b) \sim (e, f)$.

Therefore, $\sim$ is an equivalence relation.

(b) Explain why a relation defined on all pairs in $\mathbb{Z} \times \mathbb{Z}$ by the same rule as $\sim$ would not be an equivalence relation.

We now have $(1,3) \sim (0,0)$ and $(0,0) \sim (1,2)$. If $\sim$ were a equivalence relation, then by transitivity, $(1,3) \sim (1,2)$, which is false because $2 = 1 \cdot 2 \neq 3 = 3 \cdot 3$.

(c) Give a simple description of the equivalence class of $(0,1)$.

$(a,b) \sim (0,1)$ iff $a \cdot 1 = b \cdot 0$ iff $a = 0$. Thus, the equivalence class is $\{(0,b) | b \in \mathbb{Z} - \{0\}\}$.

(d) Define the *rational sum*, $+$, of two pairs in P by the rule

$$(a,b) + (c,d) ::= (ad + bc, bd)$$

Show that for pairs $r, s, t \in P$, if $r \sim s$ then $r + t \sim s + t$. This says that "$\sim$ is a congruence with respect to the rational sum operation on P."

Let $r = (r_1, r_2)$, $s = (s_1, s_2)$ and $t = (t_1, t_2)$. If $r \sim s$, then $r_1 s_2 = r_2 s_1$. So, $r + t = (r_1 t_2 + r_2 t_1, r_2 t_2)$ and $s + t = (s_1 t_2 + s_2 t_1, s_2 t_2)$. Note that

$$\begin{aligned}
(r_1 t_2 + r_2 t_1)(s_2 t_2) &= r_1 t_2 s_2 t_2 + r_2 t_1 s_2 t_2 \\
&= (r_1 s_2) t_2 t_2 + (r_2 s_2) t_1 t_2 \\
&= (r_2 s_1) t_2 t_2 + (r_2 s_2) t_1 t_2 \quad \text{since } r \sim s. \\
&= (r_2 t_2)(s_1 t_2 + s_2 t_1)
\end{aligned}$$

Therefore, $r + t \sim s + t$.

(e) Define the *rational product*, $\times$ of two pairs in P by the rule

$$(a,b) \times (c,d) ::= (ac, bd)$$

Prove that $\sim$ is a congruence with respect to the rational product.

Define r, s, t as above. We want to show that if $r \sim s$, then $r \times t \sim s \times t$. Since $r \sim s$, we have $r_1 s_2 = r_2 s_1$. Now, $r \times t = (r_1 t_1, r_2 t_2)$ and $s \times t = (s_1 t_1, s_2 t_2)$. Now,

$$(r_1 t_1)(s_2 t_2) = (r_1 s_2)(t_1 t_2) = (r_2 s_1)(t_1 t_2) = (s_1 t_1)(r_2 t_2).$$

So, $r \times t \sim s \times t$.

(f) Define the binary relation of *rational inequality*, $\preceq$, on P by

$$(a,b) \preceq (c,d) \quad \text{iff} \quad abd^2 \leq cdb^2.$$

Prove that $r \preceq s$ and $s \sim t$ implies that $r \preceq t$.

Define r, s, t as above. $r \sim t$ implies $r_1 t_2 = r_2 t_1$. Since $r \preceq s$,

$$\begin{aligned}
& r_1 r_2 s_2^2 \leq s_1 s_2 r_2^2 \\
\Longrightarrow\quad & r_1 r_2 s_2^2 t_2^2 \leq s_1 s_2 r_2^2 t_2^2 \quad \text{since } r_2^2 > 0 \\
\Longrightarrow\quad & (r_1 t_2) r_2 s_2^2 t_2 \leq s_1 s_2 r_2^2 t_2^2 \\
\Longrightarrow\quad & (r_2 t_1) r_2 s_2^2 t_2 \leq s_1 s_2 r_2^2 t_2^2 \\
\Longrightarrow\quad & t_1 r_2^2 s_2^2 t_2 \leq s_1 s_2 r_2^2 t_2^2 \\
\Longrightarrow\quad & t_1 t_2 s_2^2 \leq s_1 s_2 t_2^2 \quad \text{since } r_2^2 > 0
\end{aligned}$$

Therefore, $r \preceq t$.

(g) Prove that if $(0,1) \preceq r$ and $s \preceq t$, then $r \times s \preceq r \times t$, for all $r, s, t \in P$.

Define r, s, t as above. $(0,1) \preceq r$ implies $0 \leq r_1 r_2$. Since $s \preceq t$, we have $s_1 s_2 t_2^2 \leq t_1 t_2 s_2^2$. Thus,

$$(r_1 r_2)(r_2^2) s_1 s_2 t_2^2 \leq (r_1 r_2)(r_2^2) t_1 t_2 s_2^2 \Longrightarrow (r_1 s_1)(r_2 s_2)(t_2 r_2)^2 \leq (r_1 t_1)(r_2 t_2)(r_2 s_2)^2$$

Therefore, $r \times s \preceq r \times t$.

(h) Let $[p]_\sim$ be the equivalence class of $p \in P$ under $\sim$. Define $[p]_\sim \leq_\sim [q]_\sim$ iff $p \preceq q$. Prove that $\leq_\sim$ is a partial order on the $\sim$ equivalence classes.

In part (f), we showed that $r \preceq s$ and $s \sim t$ implies that $r \preceq t$. Using an identical argument, we can show that $r \preceq t$ and $r \sim s$ implies that $s \preceq t$. Therefore, $\preceq$ is preserved for related elements. In other words, the binary relation, $\leq_\sim$ is well-defined. With equivalence classes, if $r \sim r'$, then $[r]_\sim = [r']_\sim$. For $\leq_\sim$ to be well-defined, it cannot depend upon the representation of the equivalence class (i.e. it can't matter whether we use $[r]_\sim$ or $[r']_\sim$). By part (f), it follows that if $r \sim r'$ and $[r] \sim\leq_\sim [t]_\sim$, then $[r']_\sim \leq_\sim [t]_\sim$. Similarly, we can show if $t \sim t'$ and $[r] \sim\leq_\sim [t]_\sim$, then $[r]_\sim \leq_\sim [t']_\sim$. Thus, $\leq_\sim$ does not depend upon our choice of r (or t) for the representation. Hence, it is well-defined.

To show a partial order, we need to show reflexive, antisymmetric, and transitive.

- **Reflexive** Observe that $r \preceq r$. $(r_1 r_2^3 \leq r_1 r_2^3)$ Thus, $[r]_\sim \leq_\sim [r]_\sim$ and $\leq_\sim$ is reflexive.
- **Antisymmetric** If $[r]_\sim \leq_\sim [s]_\sim$ and $[s]_\sim \leq_\sim [r]_\sim$, then $r \preceq s$ and $s \preceq r$. Therefore,

$$\begin{aligned} & r_1 r_2 s_2^2 \leq s_1 s_2 r_2^2 \text{ and } s_1 s_2 r_2^2 \leq r_1 r_2 s_2^2 \\ \Longrightarrow \quad & r_1 r_2 s_2^2 = s_1 s_2 r_2^2 \\ \Longrightarrow \quad & r_1 s_2 = s_1 r_2 \\ \Longrightarrow \quad & r \sim s \end{aligned}$$

  Therefore, $[r]_\sim = [s]_\sim$ and $\leq_\sim$ is anti-symmetric.
- **Transitive** If $[r]_\sim \leq_\sim [s]_\sim$ and $[s]_\sim \leq_\sim [t]_\sim$, then $r \preceq s$ and $s \preceq t$. Therefore,

$$\begin{aligned} & r_1 r_2 s_2^2 \leq s_1 s_2 r_2^2 \text{ and } s_1 s_2 t_2^2 \leq t_1 t_2 s_2^2 \\ \Longrightarrow \quad & r_1 r_2 s_2^2 t_2^2 \leq s_1 s_2 r_2^2 t_2^2 \text{ and } s_1 s_2 t_2^2 r_2^2 \leq t_1 t_2 s_2^2 r_2^2 \\ \Longrightarrow \quad & r_1 r_2 s_2^2 t_2^2 \leq t_1 t_2 s_2^2 r_2^2 \\ \Longrightarrow \quad & r_1 r_2 t_2^2 \leq t_1 t_2 r_2^2 \text{ since } s_2^2 > 0 \end{aligned}$$

  Therefore, $r \preceq t$, and $[r]_\sim \leq_\sim [s]_\sim$. Thus, $\leq_\sim$ is transitive.

Thus, we have shown $\leq_\sim$ is a partial order.