# Conflict Final Exam

This is an OPEN BOOK Final: you may refer to books, handouts, or your own course notes. (Calculators are not allowed.) You may use in your problem solutions any of the mathematical results given in lectures, recitations, or the course text. There are twelve problems of roughly equal weight to complete within three hours.

Please do not discuss the problems with other sudents in the class before the regular final exam which is scheduled for Friday morning.

Write all solutions in the exam book. You may use the back of the pages if you need more space.

Good luck!

| Problem | Grade |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| 12 | |
| Total | |

**Name:** ______________________________

**Problem 1.** Let $A$ be a set and $R$ be a binary relation on $A$. For each of the statements in the box on the left below, indicate the number of the formula in the box on the right expressing that property. (Recall that the *composite* of two binary relations $R$ and $S$, written $S \circ R$, is the relation $\{(a, c) \mid \text{there exists some } b \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$. $R^T$ is the *transpose* of $R$, i.e. $R^T = \{(a, b) \mid (b, a) \in R\}$, and $I$ denotes the *identity* relation on $A$, i.e. $I = \{(a, a) \mid a \in A\}$.)

| | | |
|---|---|---|
| a. | $R$ is a transitive relation | |
| b. | $R$ is antisymmetric | |
| c. | $R$ is reflexive | |
| d. | $R$ is symmetric | |
| e. | $R$ is a function | |
| f. | $R$ is an equivalence relation | |
| g. | $R$ is a partial order | |

1. $I \subseteq R$
2. $R = R^T$
3. $I \subseteq R \circ R \subseteq R \subseteq R^T \cap R$
4. $R \cap R^T \subseteq I \subseteq R \circ R \subseteq R$
5. $R \circ R \subseteq R$
6. $R \circ R^T \subseteq I \subseteq R^T \circ R$
7. $R \cap R^T \subseteq I$

**Problem 2.** In this problem, $A$, $B$ and $C$ denote arbitrary sets. $f$ and $g$ are functions as specified in each part and $I$ denotes the identity function. For each statement indicate whether it is true or false. No explanation is required.

**T F** There exists an injection (one-to-one) $f : A \to B$ if and only if there exists a surjection (onto) $g : B \to A$.

**T F** If for all $f : A \to B$ and $g : B \to A$, $f \circ g = I$ implies $g \circ f = I$, then $A$ and $B$ are finite.

**T F** If $f : A \to B$ is an injection (one-to-one), then there is a $g : B \to A$, such that $f \circ g = I$.

**T F** Let $f : A \to B$ and $g : B \to C$. $f \circ g$ is a surjection implies both $f$ and $g$ are surjections.

**T F** Let $f : A \to B$ and $g : B \to A$. $f \circ g = I$ implies $f$ is a surjection.

**Problem 3.** For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with $o$, $O$, $\omega$, $\Omega$, and $\Theta$ depending on whether $f(n) = o(g(n))$, $f(n) = O(g(n))$, $f(n) = \omega(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write the strongest one. Suboptimal answers will get partial credit but incorrect answers will be penalized, so do not guess unless you are reasonably sure. As usual, $\lg n$ denotes logarithm to the base 2, and $\lg^2 n$ denotes $(\lg n)^2$.

$$8^{\lg n} = \underline{\qquad\quad}(n^{7.5})$$

$$n^n(n!) = \underline{\qquad\quad}(2n)!$$

$$2^{\lg^3 n} = \underline{\qquad\quad}n(\lg^2 n)$$

$$n^{375} = \underline{\qquad\quad}\binom{n}{375}$$

**Problem 4.** We wish to prove by induction on $n$ that the recurrence

$$T(n) = 2T(n/3) + T(n/4) + n$$

has an asymptotic growth rate of $O(n)$.

Which of the following induction hypothesis will provide a correct proof? (List all correct answers; more than one may be correct.)

$$\begin{array}{ll} (i) & T(n) \le 13n \\ (ii) & T(n) \le 4n^2 \\ (iii) & T(n) \le 10n \\ (iv) & T(n) \le 4n \\ (v) & T(n) = O(n) \\ (vi) & T(n) \le 10n + 10 \end{array}$$

Answer: ________________________

**Problem 5.** Prove that a countable union of countable sets is countable. In other words, let $A_1, A_2, \ldots$ be countable sets. Let $A = \bigcup_{n \geq 1} A_n$. Show that $A$ is countable.

**Problem 6.** Find the number of permutations of the letters $\{a, b, c, d, e, f, g, h\}$ that do not contain "be" or "dafc".

**Problem 7.** Let $G$ be an undirected graph with $k$ nodes. Prove that the number of colorings of $G$ with $n$ colors is $\Theta(n^k)$.

**Problem 8.** Each of $n$ people independently picks a number from $\{1, 2, 3, \ldots, n\}$ according to a uniform distribution. Let $X$ denote the product of the number selected.

**(a)** What is $E(X)$?

**(b)** How does your answer in part (a) compare to $n!$ ? (Use asymptotic notation.)

**(c)** What is $Var(X)$? (Hint: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$)

**Problem 9.** Solve

$$\begin{aligned} 3x + 5y &= 7 \\ 4x + 2y &= 3 \end{aligned}$$

in mod 11.

**Problem 10.** (*Two-ball roulette*) At a casino, a two-ball version of roulette is played. Players place their bets as in the regular roulette, and collect winnings for each of the balls, e.g. if a player bets on red, and only one ball lands on red he breaks even; if both balls land on red he wins his bet, and if none of the balls lands on red he loses his bet.

There are 18 red, 18 black and two green numbers in roulette. We assume that in the two-ball roulette the two balls cannot land on the same number, but they are equally likely to land on any pair of distinct numbers. For simplicity assume players only bet on red or black.

**(a)** What is the probability of winning, losing and breaking even?

**(b)** What is the expected winning on a single \$100 bet on red?

**(c)** Is the two-ball roulette more or less profitable for the casino than the standard roulette?

**Problem 11.** Consider the 5 letter words over an alphabet of 26 letters.

**(a)** How many different 5 letter words (not necessarily meaningful) are there?

**(b)** We now consider two words to be equivalent if they are permutations of each other, e.g. "asdfa" $\equiv$ "dsfaa". How many such equivalence classes of words are there?

**Problem 12.** Let $S = \{1, 2, 3, \ldots, 200\}$. Prove that for any set $A \subset S$ of 101 numbers, there are two numbers $a, b \in A$ such that $a|b$.