Problem 0-1. **(a)** Let $G = (V, E)$ be a directed graph in which each vertex has out-degree at least 1. Prove by induction that G contains a directed path of length m for all $m \in \mathbb{N}$.

(b) Use the pigeonhole principle to prove that if directed graph $G = (V, E)$ has a path with $|V|$ edges, then G contains a cycle.

Problem 0-2. For each of the following statements, (1) circle **T** or **F** to indicate whether the statement is true or false, respectively, and (2) justify your answer succinctly. The justification counts for more points than does the correct **T/F** answer. A formal proof is not necessary, but you should be able to write a one- or two-sentence explanation or picture that conveys the essential idea.

T F $f(x) = x^3 + x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

T F $f(x) = x^3 - x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

T F $\forall a, b, c \in \mathbb{N}$, if $a \mid c$ and $b \mid c$, then $ab \mid c$.

T F $\forall a, b \in \mathbb{N}$, we have $\gcd(7a + 2b, 3a + b) = \gcd(a, b)$.

T F For all countably infinite sets A and B, if A is a proper subset of B, then there is no bijection between A and B.

Problem 0-3. For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with o, O, ω, Ω, and Θ depending depending on whether $f(n) = o(g(n))$, $f(n) = O(g(n))$, $f(n) = \omega(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write the strongest one. Suboptimal answers will get partial credit but incorrect answers will be penalized, so do not guess unless you are reasonably sure. As usual, $\lg n$ denotes logarithm to the base 2. The first line is an example solution.

$$
\begin{aligned}
8^{\lg n} &= \underline{\qquad\quad}(n^3) \\
n! &= \underline{\qquad\quad}(6^n) \\
2^{(\lg^3 n)} &= \underline{\qquad\quad}(n^4) \\
\text{(for any constant } k,\text{)}\ \binom{n}{k} &= \underline{\qquad\quad}(n^k) \\
\sum_{i=1}^{n} \frac{1}{i} &= \underline{\qquad\quad}(1) \\
\sum_{i=n}^{\infty} \frac{1}{i^2} &= \underline{\qquad\quad}(1)
\end{aligned}
$$

Problem 0-4. In an $\mathbb{N}^+$-weighted undirected graph, each edge is assigned a positive integer weight. The **weighted-degree** of a node is the sum of the weights of its incident edges.

Consider the following set of weighted degrees: $1, 3, 5, 7, \ldots, 2n - 1$.

(a) Show that if n is odd, then no graph on n nodes has this set of weighted-degrees.

(b) Show that if $n > 2$ is even, then there exists a graph on n nodes with this set of weighted-degrees.
Do this by constructing such a graph for each even $n > 2$. You may do this inductively, by constructing a graph for $n+2$ from the graph already constructed for n. (Don't forget the base case.)

Problem 0-5. There are k (distinguishable) students taking an exam who must be seated in a row of n seats so that no two students are seated next to each other. How many ways can the students be seated?

Problem 0-6. Assume that the annual inflation rate is forever fixed at $p = 5\%$. Suppose Marilyn offers to pay Bobo $\$1000n$ dollars at the end of year n for every $n \geq 1$. So, at the end of year 1, Bobo gets \$1000; at the end of year 2, Bobo gets another \$2000, at the end of year 3, Bobo gets another \$3000, and so on, forever. In return, Marilyn wants an up front payment of $\$1,000,000$. Bobo figures this is a good deal since he (or his descendants) will be making an infinite amount of money in the future. Is Bobo right about it being a good deal? What is the present value of all future payments? You should not need a calculator to compute the exact answer.

Problem 0-7. Observe that

$$\begin{aligned}
1 &= 1 \\
4 &= 1+3 \\
9 &= 1+3+5 \\
16 &= 1+3+5+7 \\
25 &= 1+3+5+7+9
\end{aligned}$$

Use generating functions to derive and prove a general formula. For this problem, you *must* use generating functions.

Problem 0-8. The prosecution at OJ's trial is attempting to tie OJ to the crime by a new analysis of *blood thickness*. Blood thickness is expressed as a natural number. The average blood thickness over all humans is 100. OJ has very thick blood: its thickness is 150.

(a) Knowing only the average thickness, what bounds (upper and lower) can you prove on the fraction of humans that can have blood thickness 150?

(b) At the trial, the prosecution reveals that the standard deviation of blood thickness is 5. It is further revealed that the blood found at the scene of the crime has thickness 150. Using Chebyshev's theorem, what can you say about the fraction of humans that might have commited the crime?

Problem 0-9. If m identical balls are thrown randomly into n identical bins, what is the expected number of empty bins?

Problem 0-10. How many $n \times n$ (0-1)-matrices have no all-0 columns?

Problem 0-11. Let G be an undirected graph with $2n$ vertices. Argue carefully that if G contains no triangles, then G contains at most n^2 edges, and that this bound is tight.

Problem 0-12. Let X be a set of size n, and let $S_1, S_2, \cdots, S_b$ be a collection of (not necessarily distinct) subsets of X, all of size l. Suppose there exist integers t and λ such that if Y is any subset of X of size t, then $Y \subseteq S_i$ for exactly λ of the sets S_i. (Of course, $1 \le t \le l$ and $1 \le \lambda \le b$.) Suppose that R is a subset of X of size k, for some k, where $1 \le k \le t$. Argue that R is a subset of exactly $\lambda \binom{n-k}{t-k} / \binom{l-k}{t-k}$ of the subsets S_i.

Problem 0-13. Prove that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \qquad \text{for } n = 2, 3, 4, \ldots$$

Problem 0-14. Give asymptotic upper bounds for $T(n)$ in each of the following recurrences. Assume that $T(n)$ is constant for $n \le 2$. Make your bounds as tight as possible.

(a) $T(n) = 2T(n/2) + n^3$

(b) $T(n) = T(9n/10) + n$

(c) $T(n) = 16T(n/4) + n^2$

(d) $T(n) = 7T(n/3) + n^2$

(e) $T(n) = 7T(n/2) + n^2$

(f) $T(n) = 2T(n/4) + \sqrt{n}$

(g) $T(n) = T(n-1) + n$

Problem 0-15. Suppose you are offered a betting game in which you keep flipping a fair coin until you get a head and if you flip the coin k times without getting a head, you win $\$2^k$. If the game costs \$1 to play, what is your expected return (net winnings)?
Would you be willing to play this game? Would you be willing to play this game if it cost \$1000?

Problem 0-16. When designing VLSI circuits, a good rule of them is that the probability that the circuit will work is e^{-pA} for some constant p, where A is the area of the circuit. Give a justification for this rule under the assumption that every unit of area on the chip fails independently with probability p and that the circuit will fail if any unit of area fails.

Problem 0-17. Recall the Gambler's Ruin problem from lecture. Assume the game is fair—that is, $p = 1/2$. Suppose you have a goal of reaching $\$T$. Let P_n be the probability that, starting with $\$n$, where $n \le T$, you reach your goal before going broke. In lecture we showed that $P_n = n/T$.

Suppose you start with $\$n$ and continue to play until you are broke. Let random variable M be the maximum number of dollars you ever have. Determine the expected value of M.