Massachusetts Institute of Technology — Quiz 3
6.042J/18.062J: Mathematics for Computer Science — December 6, 1995
Professors David Karger and Tom Leighton

Quiz 3

- Put your name on the top of each page—the pages of this booklet will be separated for grading.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress. If you are unsure about something then state your assumptions.
- Show your work, as partial credit will be given. Full credit will not be given for unjustified answers. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. **Be neat**.
- This quiz ends at 9 P.M. It contains seven problems. You will get 2 points for writing your name and circling your tutorial TA clearly.
- GOOD LUCK!

Problem	Points	Grade
name	2	
1	15	
2	15	
3	20	
4	20	
5	25	
6	10	
7	15	
8	15	
Total	137	

Please circle your tutorial TA: Matt Andrew Jon Miklos

Name:________________________

Problem 1. [15 points] Suppose I roll three independent standard dice and subtract the smallest value from the largest value.

(a) [5 points] What is the probability distribution for the resulting difference? (i.e. give the probability of getting the result k for all k)

(b) [5 points] What is the mean of the result?

(c) [5 points] What is the variance of the result?

Problem 2. [15 points] Suppose I roll ten independent standard dice, and consider the following three events:

- the sum of the first six is congruent to 1 modulo 3.
- the sum of the last six is congruent to 1 modulo 3.
- the product of all the dice is a power of two.

(a) [9 points] Are these events pairwise independent? Why, or why not?

(b) [6 points] Are they mutually independent? Why, or why not?

Problem 3. [20 points] For some number k, there is a bag with $1000 - k$ regular nickels and k two-headed nickels. Experiment X1 consists of pulling a coin out of the bag at random and flipping it once before putting it back in.

(a) [5 points] You perform X1 once. What is the probability you get 1 head?

(b) [5 points] You repeat X1 10 times (i.e. you pull a coin out, flip it and put it back in, repeating the whole process 10 times). What is the probability you get 10 heads?

(c) [10 points] What is the smallest value of k so that when you repeat X1 10 times, the most likely number of heads is 6? (Justify your answer)

Problem 4. [20 points] There is a bag with 1000 nickels, and we assign each one a heads probability 1/3 with probability 2/3, and a heads probability 2/3 with probability 1/3. Experiment X2(k) consists of pulling a coin out of the bag at random and then flipping the same coin k times before putting it back.

(a) [5 points] You perform X2(10). What is the probability you get 10 heads?

(b) [5 points] You perform X2(10). Given that you get 10 heads what is the probability that the heads probability of the coin you pulled out is equal to 2/3?

(c) [10 points] You perform X2(10). Given that you get 10 heads what is the probability that another 10 flips with the coin you pulled out will all come up heads?

Problem 5. [25 points] A space station can tolerate up to k meteor hits in its lifetime. Each year there is a probability p that it gets hit by exactly one meteor and probability $1-p$ that it gets hit by none. Each year there is also a probability of $q < p$ that each of the current hits will be repaired, provided that it happened within the last ten years. Finally, there is a probability w that a nuclear war will destroy the entire world, and the space station too, and a probability u that a UFO will arrive and restore the station to its original condition.

(a) [10 points] What is the probability $P(n)$ that the space-station survives exactly n years, i.e. the k'th meteor hits the space-station in year n?

(b) [8 points] Suppose there are r different types of meteors, and the space station is only destroyed if it gets hit by all r types within a period of y years. Assuming UFO's but no risk of a nuclear war, what is the expected lifetime of a space station?

(c) [7 points] Suppose the probability of a hit in year m is $1/(m+1)$. What is the probability that it gets hit sometime in the first n years?

Problem 6. [10 points] Suppose I have an urn with N balls of n different colors. We draw a ball at random, put it back, and then draw balls repeatedly until we again get a ball of this first color. What is the expected number of balls drawn?

Problem 7. [15 points] Adam and Eve marry and begin having children. Each child is a boy or a girl independently with probability 1/2, and Adam and Eve stop having children once they have two children of the **same** sex. Let c be the expected number of children they have.

(a) [5 points] What is the value of c?

(b) [10 points] Among the children in a generation, as many children as possible marry each other and have children according to the same rule. What is the expected number of descendents in year n?

Problem 8. [15 points] Recall the hashing problem from tutorial. Show that with high probability, the maximum number of items hashed to one location is $\Omega(\log n / \log \log n)$.