We consider the problem of hashing records. Each record contains a key. It may contain additional information. We have a hash-table with n slots numbered $1 \ldots n$. When we get a new record, a hash function is applied to its key to obtain a number i between 1 and n, and the record is then stored in the ith slot of the hash-table. If multiple records get hashed to the same hash-table location, then they are stored as a linked list.
The hash functions are chosen to be "random". For the purposes of this question, we assume that a key is hashed uniformly among the number 1 to n (i.e. with equal probability).
In the two problems below we consider the complexity of searching a hash-table. Assume the hash-table contains n records.

1. *Expected max search time*

    - Let X_i be the number of records in the i'th location of the hash-table. Prove that

    $$Pr(X_i \geq k) \leq \binom{n}{k} \left(\frac{1}{n}\right)^k$$

    - Prove that $\binom{n}{k} \leq (\frac{ne}{k})^k$ and hence conclude that

    $$Pr(X_i \geq k) \leq (\frac{e}{k})^k$$

    - If $k \geq (2e \ln n / \ln \ln n)$ then show that

    $$Pr(X_i \geq k) \leq n^{-2}$$

    - Prove that the expected maximum search time for a record is not very large by showing that $E(max_i X_i) = O(\ln n / \ln \ln n)$.

2. *Expected average search time*

    - Let X_i be the number of records in the i'th location of the hash-table. Prove that the expected average search time is

    $$E((\frac{1}{n})(\sum_i X_i(X_i + 1)/2)).$$

    - Let Y_j be the indicator variable which is 1 if the j'th record is hashed to the i'th location. Then $X_i = \sum_j Y_j$. Using the fact that the Y_j are independent prove that $E(X_i^2) = 2 - \frac{1}{n}$.
    - Show that the expected average search time is $1.5 - \frac{1}{2n}$.