Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors David Karger and Tom Leighton

FINAL EXAM
December 19, 1995

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- This quiz is closed book. You may use three handwritten crib sheets. No calculators allowed.
- Put your name on the top of each page.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem.
- Show your work, as partial credit will be given. Full credit will not be given for unjustified answers. **Be neat**.

Problem	Points	Grade
name	2	
1	15	
2	15	
3	20	
4	15	
5	10	
6	20	
7	20	
8	10	
9	15	
10	15	
11	15	
12	15	
13	20	
Total	207	

Name:_____

Please circle your tutorial TA's name: Matt Andrew Jon Miklos

Problem 1. [15 points] Prove **using induction** that the sum of the cubes of three consecutive positive integers is divisible by 9.

Problem 2. [15 points] For each of the following statements, (1) circle **T** or **F** to indicate whether the statement is true or false, respectively, and (2) justify your answer succinctly. The justification counts for more points than does the correct **T**/**F** answer. A formal proof is not necessary, but you should be able to write a one- or two-sentence explanation or picture that conveys the essential idea.

T **F** [3 points] $\forall X \subseteq \mathbb{R}$, if X is uncountable, then $\mathbb{R} - X$ is countable.

T **F** [3 points] $f(x) = |x|$ is an injection from $\mathbb{R}$ to $\mathbb{R}$.

T F [3 points] $f(x) = x^{1995}$ is a surjection from $\mathbb{R}$ to $\mathbb{R}$.

T F [3 points] For all finite sets A and B, if A is a proper subset of B, then there is no bijection between A and B.

T F [3 points] Let a_n be the probability that you pick the maximum on picking a number uniformly at random from an array of n distinct numbers. Assume $a_0 = 0$. Then the OGF for a_n is $x/(1-x)^2$.

Problem 3. [20 points] Short answer questions. Justify your answer succinctly.

(a) [5 points] What is the coefficient of x^3 in the binomial expansion of $(1-x)^{-\frac{1}{2}}$?.

(b) [5 points] Evaluate the sum

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x^{i-j} y^{i+j}$$

(c) [10 points] Find the number of three letter words over the alphabet a,b,...,z that do not contain "ab" or "ba".

Problem 4. [15 points] Let A and B be two equivalence relations on the set $S = \{1, 2, \ldots, n\}$. Prove that $A \bigcap B$ is an equivalence relation too. (Recall that a relation on a set S is a subset of $S \times S$.)

Problem 5. [10 points] For each of the following pairs of functions $f(n)$ and $g(n)$, fill in the blank with o, O, ω, Ω, and Θ depending on whether $f(n) = o(g(n))$, $f(n) = O(g(n))$, $f(n) = \omega(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write the strongest one. Suboptimal answers will get partial credit but incorrect answers will be penalized, so do not guess unless you are reasonably sure. As usual, $\lg n$ denotes logarithm to the base 2. Each part is worth 2 points.

$$10^{\lg n} \quad = \quad _____(n^{\lg 10})$$

$$(n!)^2 \quad = \quad _____(n^n)$$

$$2^{(\lg^3 n)} \quad = \quad _____(n^{4 \lg \lg n})$$

$$\text{(for any constant } k\text{,) } \binom{n}{k} \quad = \quad _____(n^k)$$

$$\sum_{i=1}^{n} \frac{1}{i^5} \quad = \quad _____(1)$$

Problem 6. [20 points]

(a) [10 points] Let

$$T(n) = 2T(\lfloor n/4 \rfloor) + T(\lfloor n/3 \rfloor) + 2n$$

for all positive integers n and let $T(0) = 0$. Prove **by induction** that $T(n) = O(n)$.

(b) [10 points] Solve

$$T(n) = 5T(n-1) - 4T(n-2)$$

for all $n \geq 2$. $T(0) = 3$ and $T(1) = 6$.

Problem 7. [20 points]

(a) [10 points] How many ways can you arrange 39 identical black balls and 13 identical white balls in a row, so that no two white balls are adjacent?

(b) [10 points] A standard deck of 52 cards is shuffled completely (i.e. any of the permutations are equally likely.) What is the **probability** that no two spades are adjacent in the shuffled deck?

Problem 8. [10 points] A pitcher's record is 21 wins and 4 losses. **Prove** that he must have enjoyed a winning streak of 5 games.

Problem 9. [15 points] 10% of the theorems in the lecture notes contain typos. For any theorem, it is the case that if you ask a TA whether it has a typo the probability that her answer is correct is 0.8.

(a) [7 points] Given a theorem selected uniformly at random, if a TA tells you that it has a typo, what is the probability that it does?

(b) [8 points] Suppose TAs answer independently from each other. If 2 TAs tell you a theorem has a typo, what is the probability that it does?

Problem 10. [15 points] An integer X is drawn from the set $1, 2, \ldots, n$ uniformly at random.

(a) [5 points] Prove that the probability that X is divisible by k is $(1/n)(\lfloor n/k \rfloor)$.

(b) [10 points] Prove that the expected number of divisors of X is $\Theta(\log_e n)$

Problem 11. [15 points]

(a) [5 points] Let G be a graph with E edges. If each vertex is colored red or blue independently with equal probability then what is the expected number of edges with differently colored endpoints.

(b) [10 points] Let $N(i,j), 1 \leq i,j \leq n$ represent the number of times person i calls person j, per month. Calls within an area code cost L, and calls between area codes cost D. If the phone company randomly assigns each person to one of k area codes, then what is its expected monthly revenue?

Problem 12. [15 points] 100 students take an exam. The total number of points for the exam is 207.

(a) [7 points] If the mean on the exam is 60 then what is a nontrivial upper bound on the number of students who scored **more than** 120?

(b) [8 points] If the mean on the exam is 60 and the variance is 72 then what is the best upper bound you can give on the number of students who scored **more than** 120?

Problem 13. [20 points] An exam has 100 true/false questions and 100 multiple choice questions. Each true/false question is worth 1 point while each multiple choice question is worth 2 points. The maximum score is 300. There is a student who answers at random. She is correct with probability 1/2 on each true/false question and with probability 1/4 on each multiple choice question.

(a) [3 points] What is the expected score of the student?

(b) [7 points] What is the standard deviation of the student's score?

(c) [10 points] What is the best upper bound you can give on the probability that the student scores more than 272 points?