Quiz 3 Solutions

Problem 3-1. Suppose I roll two independent standard dice and subtract the smaller value from the larger value.

(a) What is the probability distribution for the resulting difference? (i.e. give the probability of getting the result k for all k)

Let R be the random variable that is the larger die roll minus the smaller die roll. The easiest way to see the probability distribution of R is to make a table of the 36 outcomes and see what the value of the larger minus the smaller is.

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Since these entries are all equally likely, the probability $R = k$ is just the number of entries that are k divided by 36.

$$
\begin{aligned}
Pr(R < 0) &= 0 \\
Pr(R = 0) &= \frac{6}{36} = \frac{1}{6} \\
Pr(R = 1) &= \frac{10}{36} = \frac{5}{18} \\
Pr(R = 2) &= \frac{8}{36} = \frac{2}{9} \\
Pr(R = 3) &= \frac{6}{36} = \frac{1}{6} \\
Pr(R = 4) &= \frac{4}{36} = \frac{1}{9} \\
Pr(R = 5) &= \frac{2}{36} = \frac{1}{18} \\
Pr(R > 5) &= 0
\end{aligned}
$$

(b) What is the mean of the result?

The mean is

$$
\begin{aligned}
E[R] &= \sum_{k=0}^{5} kPr(R=k) \\
&= (1)\frac{6}{36} + (2)\frac{10}{36} + (3)\frac{8}{36} + (4)\frac{6}{36} + (5)\frac{4}{36} + (6)\frac{2}{36} \\
&= \frac{70}{36} \\
&= \frac{35}{18}
\end{aligned}
$$

(c) What is the variance of the result?
The variance is $E[R^2] - (E[R])^2$.

$$
\begin{aligned}
E[R^2] &= \sum_{k=0}^{5} k^2 Pr(R=k) \\
&= (1^2)\frac{6}{36} + (2^2)\frac{10}{36} + (3^2)\frac{8}{36} + (4^2)\frac{6}{36} + (5^2)\frac{4}{36} + (6^2)\frac{2}{36} \\
&= \frac{210}{36} \\
&= \frac{105}{18} \\
E[R] &= \frac{35}{18} \quad \text{(from part b)}
\end{aligned}
$$

So the variance is

$$
\frac{105}{18} - \left(\frac{35}{18}\right)^2 = \frac{665}{324}
$$

Problem 3-2. Suppose I roll two independent standard dice, and consider the following three events:

- the first die is odd
- the second die is odd
- the sum of the dice is odd

(a) Show these events are pairwise independent.
Let A, B and C denote the first second and third events respectively.
The first and the second events are independent by definition: $P(A) = P(B) = 1/2 = P(A|B)$.
The first and the third events are independent because given the first die is odd the sum is odd if and only if the second die is even, an event which happens with probability $1/2$. Thus $P(C|A) = 1/2$. On the other hand, the unconditional probability of the third event is $1/2$, too. Indeed: $P(C) = P(A)P(\bar{B}) + +P(\bar{A})P(B) = 1/4 + 1/4 = 1/2$. Thus $P(C) = P(A|C)$ as claimed.
An identical argument shows that B and C are independent.

(b) Are they mutually independent? Why, or why not?
A, B and C are not mutually independent since $P(A)P(B)P(C) = (1/2)^3 = (1/8)$ whereas $P(A \cap B \cap C) = 0$. (Indeed: if both dice are odd, the sum is necessarily even).

Problem 3-3. There is a bag with 999 regular nickels and 1 two-headed nickel. Experiment X1 consists of pulling a coin out of the bag at random and flipping it once before putting it back in.

(a) You perform X1 once. What is the probability you get 1 head?
The probability is
$$\frac{999}{1000} \cdot \frac{1}{2} + \frac{1}{1000} \cdot 1 = \frac{1001}{2000}.$$

(b) You repeat X1 10 times (i.e. you pull a coin out, flip it and put it back in, repeating the whole process 10 times). What is the probability you get 10 heads?
Since the experiments are independent, the probability is
$$\left(\frac{1001}{2000}\right)^{10}.$$

(c) You repeat X1 10 times. What is the most likely number of heads you get, i.e. for which k is the probability of getting k heads maximized? (Justify your answer)
The distribution of heads is binomial, with success probability $p = \frac{1001}{2000}$. Thus, by taking ratios of $Pr(X = k-1)$ to $Pr(k)$ for increasing values of k as in class, we see that the probability increases for $k \leq np = \frac{5005}{1000}$, and then decreases. Thus the most likely outcome is 5.

Problem 3-4. There is a bag with 999 regular nickels and 1 two-headed nickel. Experiment X2(k) consists of pulling a coin out of the bag at random and then flipping the same coin k times before putting it back.

(a) You perform X2(10). What is the probability you get 10 heads?
Let A be the event you draw the two-headed nickel and B be the event you get 10 heads in a row. Then,

$$\begin{aligned} Pr(B) &= Pr(B \wedge A) + Pr(B \wedge \overline{A}) \\ &= Pr(B|A)Pr(A) + Pr(B|\overline{A})Pr(\overline{A}) \end{aligned}$$

Clearly, $Pr(A) = 1/1000$ and $Pr(B) = 999/1000$. If you get the two-headed nickel, you will get all heads, so $Pr(B|A) = 1$. The probability of getting ten heads with the fair nickel is 2^{-10}. Thus, $Pr(B|\overline{A}) = 2^{-10}$. Therefore,
$$Pr(B) = \frac{1}{1000} + \frac{999}{1000}2^{-10} = \frac{2023}{1024000}$$

(b) You perform X2(10). Given that you get 10 heads what is the probability that you pulled out the two-headed coin?

$$Pr(A|B) = \frac{Pr(A \wedge B)}{Pr(B)}$$

We have $Pr(B)$ from (a). $Pr(A \wedge B) = 1/1000$, since if we draw the two-headed nickel, we will get all heads.

Thus,

$$Pr(A|B) = \frac{1/1000}{2023/1024000} = \frac{1024}{2023}$$

(c) You perform X2(10). Given that you get 10 heads what is the probability that another 10 flips with the coin you pulled out will all come up heads?

Let C be the event that the next 10 flips are heads. There are two ways to do this problem.

Method 1:

Note that the event $C \wedge B$ is equivalent to getting 20 heads in a row.

$$\begin{aligned} Pr(C|B) &= \frac{Pr(C \wedge B)}{Pr(B)} \\ &= \frac{Pr(20 \text{ heads})}{Pr(10 \text{ heads})} \\ &= \frac{\frac{1}{1000} + \frac{999}{1000}2^{-20}}{\frac{1}{1000} + \frac{999}{1000}2^{-10}} \\ &= \frac{1,049,575}{2,071,552} \end{aligned}$$

Method 2:

$$\begin{aligned} Pr(C|B) &= Pr(C \wedge A|B) + Pr(C \wedge \overline{A}|B) \\ &= Pr(C|A \wedge B)Pr(A|B) + Pr(C|\overline{A} \wedge B)Pr(\overline{A}|B) \end{aligned}$$

In (b), we determined $Pr(A|B)$. $Pr(\overline{A}|B) = 1 - Pr(A|B)$.

Now, $Pr(C|A \wedge B) = 1$, since the coin is two headed. $Pr(C|\overline{A} \wedge B) = 2^{-10}$, since the flips are independent.

Therefore,

$$Pr(C|B) = \frac{1024}{2023} + \left(1 - \frac{1024}{2023}\right) 2^{-10} = \frac{1,049,575}{2,071,552}$$

Problem 3-5. A space station can tolerate up to k meteor hits in its lifetime. Each year there is a probability p that it gets hit by exactly one meteor and probability $1-p$ that it gets hit by none.

(a) What is the probability $P(n)$ that the space-station survives exactly n years, i.e. the k'th meteor hits the space-station in year n?

In order for the space station to survive exactly n years, we have to have the k^{th} meteor hit at year n. This can be rephrased in terms of two independent events. Let S be the event that the space station survives exactly n years. Let M be the number of meteor strikes in years $1, \ldots, n-1$. Let A be the event that a meteor hits in year n. Then S occurs only if the event $M = k$ occurs and also the event A occurs. Since the events A and $M = k$ are independent, we see that

$$\Pr[S] = \Pr[A] \cdot \Pr[M = k]$$

Of course, M has the binomial distribution, meaning

$$\Pr[M = k] = \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)}$$

Therefore, since $\Pr[A] = p$, we have

$$\Pr[S] = p \cdot \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}$$

(b) For a given j, what is the expected time between the j^{th} and $(j+1)^{st}$ meteor hits?

Let W_j be the year the j^{th} meteor hits ($W_0 = 0$), and let $X_j = W_{j+1} - W_j$ be the time between the j^{th} and $(j+1)^{st}$ meteor hits. As discussed in tutorial regarding the coupon collector problem, X_j is independent of W_j and has the negative binomial distribution with parameter p, so

$$E[X_j] = 1/p.$$

(c) What is the expected life of the space-station? (You will get partial credit for a non-closed form solution, but do not waste time evaluating hairy sums.)

By the definitions of the previous section, the time we care about is $W_k = \sum_{j=0}^{k-1} X_j$. It follows by linearity of expectation that

$$\begin{aligned} E[W_k] &= E[\sum_{j=0}^{k-1} X_j] \\ &= \sum_{j=0}^{k-1} E[X_i] \\ &= \sum_{j=0}^{k-1} 1/p \\ &= k/p \end{aligned}$$

Problem 3-6. Suppose I have an experiment with n possible outcomes, of which the i^{th} happens with probability p_i. Suppose I perform the experiment once to establish a "goal outcome." Now I repeat the experiment over and over until the "goal outcome" occurs again. Not including the first experiment, what is the expected number of experiments I will perform? (Justify your answer, and do not make any assumptions about the p_i's)

Let G be the goal outcome, and let X be the number of experiments that are performed after the first. From the problem definition, we see that

$$\Pr[G = i] = p_i.$$

Now suppose that $G = i$. Then we start performing experiments until the outcome i occurs. Since the probability of outcome i is p_i, the number of experiments X has the negative binomial distribution with parameter p_i. The expected value of this distribution is $1/p_i$. That is,

$$E[X \mid G = i] = 1/p_i.$$

It follows from the theorem for conditional expectations that

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} E[X \mid G = i] \cdot \Pr[X = i] \\ &= \sum_{i=1}^{n} \frac{1}{p_i} \cdot p_i \\ &= \sum_{i=1}^{n} 1 \\ &= n \end{aligned}$$

Problem 3-7. Adam and Eve marry and begin having children. Each child is a boy or a girl independently with probability 1/2, and Adam and Eve stop having children once they have two children of the **same** sex. Let c be the expected number of children they have.

(a) What is the value of c?

Suppose by symmetry the first child is a boy. Then the second child is a boy with probability $\frac{1}{2}$, in which case they stop; otherwise, by the pigeonhole principle (two genders, three children), they will stop on their third child. Thus the expected number of children is 2.5.

(b) Suppose each descendant grows up, marries a non-descendant of Adam and Eve, and follows the same child bearing rules as Adam and Eve. What is the expected **total** number of descendants of Adam and Eve up to and including the n^{th} generation, if Adam and Eve are generation 0, their children generation 1, and so on?

Since the number of children born to each couple is independent, we can apply Wald's theorem to the resulting branching process to conclude that the number of children born in generation n is $(2.5)^n$). Thus the number of descendants up to and including generation n is $\sum_{j=1}^{n}(2.5)^j$.