Problem Set 11 Solutions

Problem 11-1. **(a)** $E(X+Y) = E(X)+E(Y)$. $E(X) = 2\cdot(\frac{1}{4})+3\cdot(\frac{1}{4})+5\cdot(\frac{1}{2}) = \frac{15}{4}$. $E(Y) = 1 \cdot (\frac{2}{3}) + 4 \cdot (\frac{1}{3}) = 2$. Hence $E(X+Y) = 2 + \frac{15}{4} = \frac{23}{4}$.

(b)

X	Y	$max(X,Y)$	Pr	$max(X,Y) \cdot Pr$
2	1	2	$(\frac{1}{4})(\frac{2}{3})$	$\frac{1}{3}$
3	1	3	$(\frac{1}{4})(\frac{2}{3})$	$\frac{1}{2}$
5	1	5	$(\frac{1}{2})(\frac{2}{3})$	$\frac{5}{3}$
2	4	4	$(\frac{1}{4})(\frac{1}{3})$	$\frac{1}{3}$
3	4	4	$(\frac{1}{4})(\frac{1}{3})$	$\frac{1}{3}$
5	4	5	$(\frac{1}{2})(\frac{1}{3})$	$\frac{5}{6}$

Summing the last column we find that $E(max(X,Y)) = 4$.

(c) Since $X + Y - max(X,Y) = min(X,Y)$ we get that $E(min(X,Y)) = E(X + Y) - E(max(X,Y)) = \frac{7}{4}$.

Problem 11-2. Let B stand for boy and G for girl. Then the different possible sequences of babies are BG, GB, BBG, GGB, BBBG, GGGB, ... There are two sequences of length n for all $n \geq 2$. The probability of occurence of a sequence of length n is 2^{-n}. Hence,

$$\text{E(number of children)} = \sum_{k=2}^{\infty} 2k \cdot 2^{-k} = 3.$$

Problem 11-3. **(a)** It makes no difference how we arrange the numbers. The expected value is the same. Why? If we label the ends of the spinner we can define 2 random variables X and Y. X is the number pointed to by one end of the spinner and Y is the other number. We wish to maximize $E(X+Y)$. But since $E(X+Y) = E(X)+E(Y)$ and $E(X) = E(Y) = (2n+1)/2$, all arangements give the same expected value of $2n+1$.

(b) The best way to do this is to ensure that 1 is opposite 2, 3 is opposite 4, ..., and $2n-1$ is opposite $2n$. Let's call this arrangement A^*. This will maximize the expected value of the product.

Consider any arrangement that isn't the one described above. It therefore must contain an odd i that isn't paired up with $i+1$. Pick the smallest such i and call it k. Say k is opposite p and $k+1$ is opposite q. We know that p and q are bigger than k (by choosing k to be the smallest such number we are assured that all numbers smaller than k are "properly" paired up). Now consider the new arrangement where k is paired with $k+1$ and p is paired with q. It has a better expected product than the previous arrangement.

To prove this note that since all pairs are equally likely when we turn the spinner, it suffices to show that the sum of the products of the numbers (denoted by S) is larger in the second arrangement.

$$\begin{aligned} S_2 - S_1 &= k(k+1) + pq - kp - (k+1)q \\ &= k^2 + k + pq - kp - kq - q \\ &= k^2 + (1-p-q)k + (p-1)q \\ &= (k-q)(k-(p-1)) \end{aligned}$$

Thus $S_2 - S_1$ is greater than zero if k is smaller than both q and $p-1$. But this is true because of our choice of k. Thus the second arrangement is better.

Now we can transform any arrangement that isn't equal to A^* by repeatedly applying the following operation:

"Find the smallest odd i that isn't paired up with $i+1$. If i is paired with p and $i+1$ with q then pair i with $i+1$ and p with q"

Each time we perform this operation, we increase the expected value of the product. Thus A^* must be the best arrangement since it is the only arrangement where we cannot perform this operation.

Problem 11-4. Let's fix the n numbers we are given. We can assume that we are given a random permutation of the n numbers.

Let's define random variables $X_i (1 \le i \le n)$ by

$$X_i = \begin{cases} 1 & \text{if element } i \text{ is the maximum} \\ & \text{of } V[1], \ldots, V[i] \\ 0 & \text{otherwise} \end{cases}$$

Note that the number of times we update the current maximum is precisely $X_1 + \ldots + X_n$. Since expectation is a linear operator, we can compute $E(X_1 + \ldots + X_n)$ by finding $E(X_i)$ for all i and summing them up. Note also that since X_i is an "indicator" variable, we only have to find $P(X_i = 1)$. $X_i = 1$ if element i is the largest of $V[1], \ldots, V[i]$. In a random permutation, this happens with probability $\frac{1}{i}$. Why? If you take i distinct numbers and randomly permute them, the probability that the largest one occupies the last (or any) position is $1/i$. If we note that

$$P(X_i = 1) = \sum_{n_1,\ldots,n_i} P(X_i = 1 \mid \text{first } i \text{ elements are } n_1, \ldots, n_i) \cdot P(\text{first } i \text{ elements are } n_1, \ldots, n_i)$$

and further observe that the first term is $1/i$ we can conclude that $P(X_i = 1) = 1/i$. Thus

$$\begin{aligned} E(X) &= \sum_i P(X_i = 1) \\ &= \sum_{i=1}^{n} 1/i \\ &= H_n \approx \ln n \end{aligned}$$

Recall that H_n is the nth Harmonic number.

Problem 11-5. **(a)** $E(X - m) = E(X) - E(m) = m - m = 0$. Thus $X - m$ is a random variable with mean 0.

(b) Assume n is even.

$$\begin{aligned} E(|X - n/2|) &= 2^{-n}\left[\sum_{k=0}^{n}\binom{n}{k}|k - \frac{n}{2}|\right] \\ &= 2^{-n}\left[2\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}(\frac{n}{2} - k)\right] \\ &= 2^{-n}\left[n\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\right] + 2^{-n}\left[2\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}k\right] \\ &= 2^{-n}\left[n(2^{n-1} - (\tfrac{1}{2})\tbinom{n}{\frac{n}{2}})\right] + 2^{-n}\left[2\sum_{k=1}^{\frac{n}{2}-1}\binom{n-1}{k-1}(\frac{n}{k})k\right] \\ &= 2^{-n}\left[n(2^{n-1} - (\tfrac{1}{2})\tbinom{n}{\frac{n}{2}})\right] + 2^{-n}\left[2n\sum_{k=0}^{\frac{n}{2}-2}\binom{n-1}{k}\right] \\ &= 2^{-n}\left[n(2^{n-1} - (\tfrac{1}{2})\tbinom{n}{\frac{n}{2}})\right] + 2^{-n}\left[2n(2^{n-2} - \tbinom{n-1}{\frac{n}{2}-1})\right] \\ &= 2^{-n}\left[n(2^{n-1} - (\tfrac{1}{2})\tbinom{n}{\frac{n}{2}}) + 2n(2^{n-2} - \tbinom{n-1}{\frac{n}{2}-1})\right] \\ &= 2^{-n}\tbinom{n}{\frac{n}{2}}(\tfrac{n}{2}). \end{aligned}$$

Problem 11-6. **(a)** There are n/k groups of size k each. Let X_i be a random variable that denotes the number of tests performed in group i. X_i takes value 1 with probability $(1-p)^k$ and value $k+1$ with probability $1-(1-p)^k$. Hence the expected number of tests is $E(\sum_{i=1}^{\frac{n}{k}} X_i) = \sum_{i=1}^{\frac{n}{k}} E(X_i) = (\frac{n}{k})((1-p)^k + (k+1)(1-(1-p)^k)) = n(1-(1-p)^k + \frac{1}{k})$.

(b) The k must be chosen so that the derivative of the answer from part a) w.r.t. k is 0, i.e. $(1-p)^k \ln(1-p) + \frac{1}{k^2} = 0$. Assuming that p is very small, so that we can approximate $(1-p)^k$ by 1 and $\ln(1-p)$ by $-p$ we get that $k \sim \sqrt{\frac{1}{p}}$. And the resulting expectation is $n\sqrt{p}$.

(c) If $p = 0.01$ then the fraction of work saved is $1 - \sqrt{0.01} = 0.9$, i.e. 90% of the work is saved in such a situation.

(d) There are many possible improvements. Here is one. Consider a scheme where we blend all the soldiers' blood and perform the test. If it comes up negative we stop, else we split the soldiers into two groups and repeat on each group. If E_n is the expected number of test with n soldiers we see that it satisfies the recurrence

$$E_n = (1-p)^n + (1-(1-p)^n)2E_{\frac{n}{2}}.$$

It is easily proven by induction that $E_n = \Theta(pn)$. Since, in an expected sense, there will pn soldiers with the disease, and the number of soldiers with the disease is a lower bound on the number of tests, hence this scheme is optimal upto constant factors.

Problem 11-7. Consider a tournament (actually a distribution over tournaments) where each game outcome is random (50-50). For any ordering (permutation) σ of the teams let the indicator random variable X_σ be 1 if σ is a valid chain and 0 otherwise. The probability of a given ordering σ being a valid chain is 2^{-n}. So, the expected number of valid chains is $E(\sum_\sigma X_\sigma) = \sum_\sigma E(X_\sigma) = 10! \cdot 2^{-9} = 7087.5$. Since this is greater than 3000 there exists some tournament which has at least 3000 valid chains.