We are distributing a new solution to quiz problem 5 to clear up some confusion caused by the previous solution.

Problem 2-5. Consider the following nuclear reaction. There is one particle of type 1 at time instant 1. Each particle of type i existing at time t undergoes fission to produce one particle of type $i+1$ and one of type $i+2$ at time instant $t+1$.

(a) Write and justify a recurrence for $f(i,t)$, the number of particles of type i at time t. (Make sure to include the base cases for your recurrence.)

We suppose the particle types are $0, 1, 2, \ldots$. Particles of types $i-2$ and $i-1$ at time $t-1$ give rise to particles of type i at time t and this is the only way that particles of type i are formed at time t. Hence we get the recurrence

$$f(i,t) = f(i-1,t-1) + f(i-2,t-1), t > 1.$$

The base case is

- $f(1,1) = 1$.
- $f(i,1) = 0, i > 1$.
- $f(1,t) = 0, t > 1$. This is because no particle of type 1 is ever created after time 1.
- $f(2,2) = 1$.
- $f(2,t) = 0, t \neq 2$. This is because no particle of type 2 is ever created after time 2.

To see why we need all these base cases, consider the recurrence. $f(i,t)$ depends on $i-1$ and $i-2$ at $t-1$. If $i = 1$ or 2, then $i - 2 < 0$, meaning we are asking for a value where f is not defined! Therefore, we have to handle $i = 1$ and $i = 2$ as base cases. Ditto for $t = 0$.

In the original solution, we gave an equally valid set of base cases following from a slightly different assumption about the problem. Namely, there are particles of type $0, 1, 2, \ldots$ and also particles of type $-1, -2, -3, \ldots$. The recurrence is the same as before. Now, however, the base case need only handle $t = 0$. For any $t > 0$, and for any i, the quantities $f(i-1,t-1)$ and $f(i-2,t-1)$ are defined, so we don't need base cases for, e.g., $i = 0$. Of course, we still need the $t = 1$ base case.

It is a interesting convenient coincidence (but one that needs to be proven!) that both solution-interpretations lead exactly the same solution.

(b) Prove by induction that $f(i,t) = \binom{t-1}{i-t}$. (You do not need to reprove identities that were proven in class.)

The proof is by induction on t. Let $P(t)$ be

$$f(i,t) = \binom{t-1}{i-t}, \forall i.$$

Base Case. $P(1)$ is true since

- $f(1,1) = \binom{0}{0} = 1$.
- $f(i,1) = \binom{0}{i-1} = 0, i > 1$.

Inductive step. Assume $P(t)$ is true. Consider $P(t+1)$.

- $i = 1$. Then from the base case we have that for $t > 1$, $f(1,t) = 0 = \binom{t-1}{1-t}$.

- $i = 2$. If $t = 2$ then from the base case we have that $f(2, 2) = 1 = \binom{1}{0}$. If $t \neq 2$ then from the base case we have that $f(2, t) = 0 = \binom{t-1}{2-t}$.
- $i > 2$.

$$\begin{aligned} f(i, t+1) &= f(i-1, t) + f(i-2, t) && \text{by the recurrence from part a)} \\ &= \tbinom{t-1}{i-1-t} + \tbinom{t-1}{i-2-t} && \text{by inductive hypothesis} \\ &= \tbinom{t}{i-1-t} && \text{by Pascal's identity} \end{aligned}$$

Hence $P(t+1)$ is true. Thus $P(t)$ is true for all t and we are done.