Quiz 2 Solutions

Problem 2-1. **(a)** Give tight (Θ) bounds for

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}}.$$

$$\begin{aligned}
\int_1^n \frac{1}{\sqrt{x}}dx \quad &\le \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \le 1 + \int_1^n \frac{1}{\sqrt{x}}dx \\
2x^{\frac{1}{2}}|_1^n \quad &\le \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \le 1 + 2x^{\frac{1}{2}}|_1^n \\
2\sqrt{n} - 2 \quad &\le \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \quad \le 2\sqrt{n} - 1 \\
\lim_{n\to\infty} \frac{2\sqrt{n}-2}{\sqrt{n}} &\le \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{\sqrt{i}}}{\sqrt{n}} \le 1 + \lim_{n\to\infty} \frac{2\sqrt{n}-1}{\sqrt{n}} \\
2 &\le \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \frac{1}{\sqrt{i}}}{\sqrt{n}} \le 2
\end{aligned}$$

That is

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \Theta(\sqrt{n}).$$

(b) Express the following in closed form

$$\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} x^p y^q.$$

(Note that the lower indices are not constants.)

$$\begin{aligned}
\sum_{p=1}^{\infty} \sum_{q=p}^{\infty} x^p y^q \quad &= \sum_{p=1}^{\infty} x^p y^p \sum_{q=p}^{\infty} y^{q-p} \\
&= \sum_{p=1}^{\infty} x^p y^p \sum_{r=0}^{\infty} y^r \\
&= \left(\sum_{p=1}^{\infty} x^p y^p\right)\left(\sum_{r=0}^{\infty} y^r\right) \\
&= \left(\tfrac{xy}{1-xy}\right)\left(\tfrac{1}{1-y}\right).
\end{aligned}$$

Problem 2-2. I have n one-dollar bills that I wish to donate to m charities.

(a) In how many ways can I do this if I don't care how much each charity gets?

Simply insert $m-1$ bars into a string of n one-dollar bills. Then the i-th charity gets the money between bars $i-1$ and i. This gives you $\binom{n+m-1}{m-1}$ ways to distribute the money.

(b) In how many ways can I do this so that each charity gets at least \$ 10?

First give 10 dollars to each charity, then distribute the remaining $n-10m$ dollars as in part a. This gives you $\binom{n+m-1-10m}{m-1} = \binom{n-9m-1}{m-1}$ ways as long as $n \geq 10m$.

Problem 2-3. In how many ways can 5 indistinguishable covered wagons, 4 indistinguishable uncovered wagons and 1 horse be arranged in a circle?

This problem is just a permutation of a multiset, except that we need to account for rotations of a circle being the same. This can be thought of either as fixing one object, say the horse, and arranging the wagons around it ($\binom{9}{5}$ ways), or counting the permutations and dividing by 10 to avoid overcounting the rotations ($\frac{1}{10}\frac{10!}{5!4!1!}$). Either way the answer is 126. The most common mistake was to forget about overcounting the rotations.

Problem 2-4. How many 6-digit decimal numbers are there that do not contain "123" or "456" as a subsequence? (for purposes of this problem numbers may have leading 0's)

Let

- A be the set of all six-digit strings, using the digits $\{0, 1, \ldots, 9\}$.
- B be the set of six-digit strings that contain the subsequence 123.
- C be the set of all six-digit strings that contain the subsequence 456.
- D be the set of all six-digit strings that do not contain 123 or 456.

We're interested in figuring out $|D|$. Since

$$D = A - (B \cup C),$$

and $B \cup C \subseteq A$, we can use the inclusion-exclusion formula:

$$|D| = |A| - |B \cup C| = |A| - |B| - |C| + |B \cap C|.$$

First, $|A| = 10^6$, since we can put any of ten digits in each of six places.

Now, a string can contain 123 in any of four positions; once this position is chosen, we have 1000 choices for the remaining positions. However, this approach counts the string 123123 twice, so the total size of B is

$$|B| = 4(1000) - 1 = 3999.$$

By the same reasoning, we get

$$|C| = 3999.$$

Finally, there are only two strings in $B \cap C$, namely 123456 and 456123. Thus,

$$|B \cap C| = 2.$$

Plugging this into our formula, we get

$$|D| = 10^6 - 3999 - 3999 + 2 = 992004.$$

(Common Mistakes: A lot of people forgot about the double-counting of 123123 *and* 456456*, and so had $|B| = |C| = 4000$. Some people also argued that since* 123 *can appear in one of only four places in the string, $|B| = 4$.)*

Problem 2-5. Consider the following nuclear reaction. There is one particle of type 1 at time instant 1. Each particle of type i existing at time t undergoes fission to produce one particle of type $i+1$ and one of type $i+2$ at time instant $t+1$.

(a) Write and justify a recurrence for $f(i,t)$, the number of particles of type i at time t. (Make sure to include the base cases for your recurrence.)

Particles of types $i-2$ and $i-1$ at time $t-1$ give rise to particles of type i at time t and this is the only way that particles of type i are formed at time t. Hence we get the recurrence

$$f(i,t) = f(i-1,t-1) + f(i-2,t-1), t > 1.$$

The base case is

- $f(1,1) = 1$.
- $f(i,1) = 0, -\infty \leq i \leq \infty, i \neq 1$.

(b) Prove by induction that $f(i,t) = \binom{t-1}{i-t}$. (You do not need to reprove identities that were proven in class.)

(Note: Let a be any nonnegative integer and b be any integer. In what follows, we use the definition that $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ when $a \geq b \geq 0$ and otherwise $\binom{a}{b} = 0$ when . Pascal's identity holds for this definition.)
The proof is by induction on t. Let $P(t)$ be

$$f(i,t) = \binom{t-1}{i-t}, \forall i.$$

Base Case. $P(1)$ is true since

- $f(1,1) = \binom{0}{0} = 1$.

- $f(i,1) = \binom{0}{i-1} = 0, -\infty \le i \le \infty, i \neq 1.$

Inductive step. Assume $P(t)$ is true. Consider $P(t+1)$.

$$\begin{aligned} f(i,t+1) &= f(i-1,t) + f(i-2,t) && \text{by the recurrence from part a)} \\ &= \binom{t-1}{i-1-t} + \binom{t-1}{i-2-t} && \text{by inductive hypothesis} \\ &= \binom{t}{i-1-t} && \text{by Pascal's identity} \end{aligned}$$

Hence $P(t+1)$ is true. Thus $P(t)$ is true for all t and we are done.

Problem 2-6. **(a)** Give tight (Θ) bounds for $T(n)$. Assume that $T(n)$ is constant for $n \le 10$.

$$T(n) = 4T(n/2) + 6n.$$

$$\begin{aligned} T(n) &= 4T(n/2) + 6n \\ &= 4^2T(n/2^2) + 6(n + n/2) \\ &= \ldots \\ &= 4^iT(n/2^i) + 6(n + n/2 + \ldots + n/2^{i-1}) \\ &= \ldots \\ &= 4^{\log_2 n}T(1) + 6(n + n/2 + \ldots + n/2^{\log_2 n \; -1}) \\ &= n^2T(1) + O(n) \\ &= \Theta(n^2) \end{aligned}$$

(b) [10 points] Rank the following functions by order of growth; that is find an arrangement $g_1 = \alpha(g_2) = \alpha(g_3) = \ldots = \alpha(g_6)$ and state whether α is o or Θ for each step.

- $n(\log_2 n)^5$
- n^3
- $n!$
- $(n!)^2$
- n^n
- $3 \cdot 8^{\log_2 n} + 4^{\log_2 n}$

$$n(\log_2 n)^5 = o(n^3) = \Theta(3 \cdot 8^{\log_2 n} + 4^{\log_2 n}) = o(n!) = o(n^n) = o((n!)^2).$$

Problem 2-7. **(a)** Let p_k be the number of ways to make change of k cents with indistinguishable pennies (1-cent coins). Give an ordinary generating function for p_k.

$$\frac{1}{1-x}.$$

(b) Let n_k be the number of ways to make change of k cents with indistinguishable nickels (5-cent coins). Give an ordinary generating function for n_k.

$$\frac{1}{1-x^5}.$$

(c) Let s_k be the number of ways to make change of k cents with indistinguishable pennies or nickels. Give an ordinary generating function for s_k.

$$\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^5}\right).$$

(d) Give an ordinary generating function for the number of ways to make change of k cents with at least 3 nickels and an odd number of pennies. (this part may be a little hard so don't knock yourself out trying to solve it.)

$$\left(\frac{x^{15}}{1-x^5}\right)\left(\frac{x}{1-x^2}\right).$$