Problem Set 9 Solutions

Problem 9-1. Consider the number of ways of dividing n similar objects into n groups - $G_1, G_2, \ldots G_n$. We can associate each such way with a non-decreasing function f in the following way: $f(1) = f(2) = \ldots = f(|G_1|) = 1; f(|G_1|+1) = f(|G_1|+2) = \ldots = f(|G_1|+|G_2|) = 2; \ldots$. It is easy to see that this is a bijection. Hence the total number of functions is $\binom{2n-1}{n-1}$.

Problem 9-2. If n is odd then it is 2^{n-1} since exactly half the sequences have more 1's than 0's. If n is even then it is $(1/2)(2^n - \binom{n}{n/2})$ since $\binom{n}{n/2}$ sequences have an equal number of 0's and 1's.

Problem 9-3. **(a)** $f(m,n) = \binom{m+n}{m}$.

(b)

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} f(m,n)x^m y^n &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{m}x^m y^n \\
&= \sum_{s=0}^{\infty}\sum_{i=0}^{s}\binom{s}{i}x^i y^{s-i} \\
&= \sum_{s=0}^{\infty}(x+y)^s \\
&= \tfrac{1}{1-x-y}
\end{aligned}$$

Problem 9-4. **(a)**

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{1}{9i^2-3i-2} &= (1/3)\left(\tfrac{1}{3i-2} - \tfrac{1}{3i+1}\right) \\
&= (1/3)\left((1-\tfrac{1}{4}) + (\tfrac{1}{4}-\tfrac{1}{7}) + \ldots\right) \\
&= 1/3
\end{aligned}$$

(b)

$$
\begin{aligned}
\int_1^{n+1} x\log x dx \quad &\le \sum_{i=1}^{n} i\log i \quad \le \int_1^n x\log x dx \\
(x^2\log x/2 - x^2/4)|_1^{n+1} \quad &\le \sum_{i=1}^{n} i\log i \quad \le (x^2\log x/2 - x^2/4)|_1^n \\
(1/2)(n+1)^2\log(n+1) - (n+1)^2/4 - 1/4 \quad &\le \sum_{i=1}^{n} i\log i \quad \le n^2\log n/2 - n^2/4 - 1/4 \\
\lim_{n\to\infty}\left(\frac{(1/2)(n+1)^2\log(n+1) - (n+1)^2/4 - 1/4}{n^2\log n}\right) \quad &\le \lim_{n\to\infty}\frac{\sum_{i=1}^{n} i\log i}{n^2\log n} \quad \le \lim_{n\to\infty}\frac{n^2\log n/2 - n^2/4 - 1/4}{n^2\log n} \\
1/2 \quad &\le \lim_{n\to\infty}\frac{\sum_{i=1}^{n} i\log i}{n^2\log n} \quad \le 1/2
\end{aligned}
$$

i.e. $\sum_{i=1}^{n} i\log i = \Theta(n^2\log n)$.

Problem 9-5. Let $P(n)$ be "$\sum_{i=0}^{n}\binom{n}{i}2^i = 3^n$."

Base Case. $P(0)$ is true since $\sum_{i=0}^{0}\binom{0}{i}2^i = 1 = 3^0$.

Inductive Step. Assume $P(n)$ is true. Then

$$
\begin{aligned}
\sum_{i=0}^{n+1}\binom{n+1}{i}2^i &= \sum_{i=0}^{n+1}\left(\binom{n}{i} + \binom{n}{i-1}\right)2^i \quad \text{by Pascal's identity} \\
&= \sum_{i=0}^{n}\tbinom{n}{i}2^i + 2\left(\sum_{i=0}^{n}\tbinom{n}{i}2^i\right) \\
&= 3^n + 2\cdot 3^n \quad \text{by inductive hypothesis} \\
&= 3^{n+1}
\end{aligned}
$$

Problem 9-6. **(a)** $f(n) = O(g(n))$ implies $\exists C, n_0, \forall n \ge n_0, f(n) \le Cg(n)$. That is, $\exists C' = (1/C), n_0, \forall n \ge n_0, g(n) \ge C'f(n)$, or $g(n) = \Omega(f(n))$.

(b) It is clear that $f(n) + o(f(n)) = \Omega(f(n))$. To see the other direction, observe that for any $g(n) = o(f(n))$, since $\lim_{n\to\infty} g(n)/f(n) = 0$ we have that $\exists n_0, \forall n \ge$

$n_0, g(n) \leq f(n)$. That is $\exists n_0, \forall n \geq n_0, f(n) + o(f(n)) \leq 2f(n)$, or $f(n) + o(f(n)) = O(f(n))$. Hence, $f(n) + o(f(n)) = \Theta(f(n))$.

Problem 9-7. $1728 = 2^6 3^3$. Let S_k be the set of numbers between 1 and 1728 divisible by k. Then the number of numbers coprime to 1728 is

$$\begin{aligned} 1728 - |S_2| - |S_3| + |S_2 \cap S_3| &= 1728 - |S_2| - |S_3| + |S_6| \\ &= 1728 - 1728/2 - 1728/3 + 1728/6 \quad = 1728(1-1/2)(1-1/3) \\ &= 576. \end{aligned}$$

Problem 9-8. **(a)** It is easy to see that $T(n) = \Omega(n)$. Using the tree method to expand the recurrence, we find that $T(n) \leq 5n + 5(1/4+4/9)n + 5(1/4+4/9)^2 n + \ldots = 5n(1 + 25/36 + (25/36)^2 + \ldots) = (180/11)n$. Hence $T(n) = O(n)$. Thus $T(n) = \Theta(n)$.

(b) The characteristic equation is $x^2 - 2x - 3$. Its roots are 3 and -1. Hence the general solution is $A(-1)^n + B3^n$. Using the initial conditions, $T(0) = T(1) = 1$ we find that $T(n) = (3/4)(-1)^n + (1/4)3^n = \Theta(3^n)$.

Problem 9-9.

$$2^{\sqrt{\log n}} = o(2^{\log n}) = o(\log n!) = o(n^3) = o(n^{\log n})$$

Problem 9-10. **(a)** Obvious.

(b) $\forall n, f(n) = 1$ since any positive number has a unique representation in base two.

(c) Follows from parts a) and b) since the generating function which has $f(n) = 1, \forall n$ is $1/(1-x)$.