Problem Set 8 Solutions

Problem 8-1. **(a)** The proof is by induction on k. Let $P(k)$ be

$$“\left(\sum_{i=1}^{k} x_i\right)^n = \sum_{n_1+n_2+\cdots+n_k=n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.”$$

Base Case. $P(2)$ is true because the binomial theorem states that

$$\left(\sum_{i=1}^{2} x_i\right)^n = \sum_{i=0}^{n} \binom{n}{i} x_1^i x_2^{n-i} = \sum_{n_1+n_2=n} \binom{n}{n_1, n_2} x_1^{n_1} x_2^{n_2}.$$

Inductive Step. Assume $P(k)$ is true. Now consider $\left(\sum_{i=1}^{k+1} x_i\right)^n$

$$= \left(\sum_{i=1}^{k} x_i + x_{k+1}\right)^n$$

$$= \left(\sum_{n'+n_{k+1}=n} \binom{n}{n', n_{k+1}} \left(\sum_{i=1}^{k} x_i\right)^{n'} x_{k+1}^{n_{k+1}}\right) \quad \text{by binomial theorem.}$$

$$= \sum_{n'+n_{k+1}=n} \sum_{n_1+n_2+\cdots+n_k=n'} \binom{n}{n', n_{k+1}} \binom{n'}{n_1+n_2+\cdots+n_k} x_1^{n_1} x_2^{n_2} \cdots x_{k+1}^{n_{k+1}} \quad \text{by inductive hypothesis}$$

$$= \sum_{n_1+n_2+\cdots+n_{k+1}=n} \binom{n}{n_1, n_2, \ldots, n_{k+1}} x_1^{n_1} x_2^{n_2} \cdots x_{k+1}^{n_{k+1}} \quad \text{by definition of the multinomial coefficient.}$$

Hence $P(k+1)$ is true. Thus $P(k)$ is true for all k and we are done.

(b) Consider the coefficient of $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$ in $\left(\sum_{i=1}^{k} x_i\right)^n$. There are n terms in $\left(\sum_{i=1}^{k} x_i\right)^n$ and the coefficient is equal to the number of ways of choosing n_1 x_1's, n_2 x_2's, …, n_k x_k's from these n terms. Thus this coefficient is equal to the number of permutations of a multiset with finite repetition counts $n_1, n_2, \ldots, n_k$ where $n_1 + n_2 + \ldots + n_k = n$. By theorem 1.1 of Notes 15 we know that this is equal to $\binom{n}{n_1, n_2, \ldots, n_k}$. Hence we are done.

Problem 8-2. **(a)** We can choose 0 or 1 or ... or n_1 copies of a_1 in our submultiset. Similarly for the elements from a_2 through a_k. This gives us a total of $\prod_{i=1}^{k}(n_i+1)$ different submultisets.

(b) $75600 = 2^4 \times 3^3 \times 5^2 \times 7$. Hence the number of divisors $= (4+1) \times (3+1) \times (2+1) \times (1+1) = 120$.

Problem 8-3. **(a)** There are 3 choices for each disk. Once a disk is assigned to a pole its position on the pole is fixed by the other disks on the pole. Hence the total number of legal positions is 3^n.

(b) For the first book there are r choices. For the next book there are $r+1$ choices since we can think of the first book as splitting the shelf it is put on into two shelves. Similarly for the third book there are $r+2$ choices. In this way we see that the total number of ways, by the product rule, is $r \times (r+1) \times (r+2) \ldots (n+r-1)$.

Problem 8-4. **(a)** For each subset of 5 scientists there must be a lock they they cannot open. Further this same lock must be openable by every other scientist because otherwise we would have a set of six scientists who will be unable to open the cabinet. Hence there must be at least $\binom{11}{5}$ locks.

(b) By the argument in part a) any scientist must be able to open all the locks associated with every subset of 5 that he/she is not a member of. Hence each scientist must carry at least $\binom{10}{5}$ keys.

Problem 8-5. **(a)** $x^3 + x^4 + 2x^5 + x^6/(1-x)$.

(b) $x^3/(1-x)^2$.

(c) $1/(1-x^2)$.

(d) $(1+x)/(1-x)^3$.

(e) $2/(1-x)^3$.

(f) $(3x - x^2)/(1-x)^3$.

(g) $1/(1-2x) + 1/(1-3x)$.

(h) e^x.

(i) $e^x/(1-x)$.

(j) $-\ln(1-x)/x$.

Problem 8-6. **(a)** $x^2 - 2x + 7 + \frac{4}{x+1} - \frac{15}{x+2}$.

(b) $(\frac{1}{4})(\frac{5}{x+1} + \frac{14}{(x+1)^2} - \frac{5}{x-1})$.

Problem 8-7. Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$. Then, we get from the recurrence that $f(x) = 2x^2 f(x) + x^3 f(x) + x + 2x^2$. Solving for $f(x)$ we get

$$f(x) = (2x^2+x)/(1-2x^2-x^3) = 1/(1+x)+(\frac{-1+\sqrt{5}}{2})/(1-\frac{1+\sqrt{5}}{2}x)+(\frac{-1-\sqrt{5}}{2})/(1-\frac{1-\sqrt{5}}{2}x).$$

We see that, c_n, the coefficient of x^n in the above generating function is

$$(-1)^n + (-1+\sqrt{5})/(2)((1+\sqrt{5})/2)^n + (-1-\sqrt{5})/(2)((1-\sqrt{5})/2)^n.$$

We now check that $a_n = c_n = (-1)^n + (-1+\sqrt{5})/(2)((1+\sqrt{5})/2)^n + (-1-\sqrt{5})/(2)((1-\sqrt{5})/2)^n$ satisfies the conditions of the problem, by induction on n. Let $P(n)$ be the statement that

" for $k \leq n, a_n = (-1)^n + (-1+\sqrt{5})/(2)((1+\sqrt{5})/2)^n + (-1-\sqrt{5})/(2)((1-\sqrt{5})/2)^n$."

Base case. It is simple to check that $P(2)$ (and therefore $P(0)$ and $P(1)$) is true.
Inductive step. Assume $P(n)$ is true. Then

$$\begin{aligned}
a_{n+1} = & \quad 2a_{n-1} + a_{n-2} && \text{by definition.} \\
& = 2\left((-1)^{n-1} + \tfrac{-1+\sqrt{5}}{2}(\tfrac{1+\sqrt{5}}{2})^{n-1} + \tfrac{-1-\sqrt{5}}{2}(\tfrac{1-\sqrt{5}}{2})^{n-1}\right) \\
& + \left((-1)^{n-2} + \tfrac{-1+\sqrt{5}}{2}(\tfrac{1+\sqrt{5}}{2})^{n-2} + \tfrac{-1-\sqrt{5}}{2}(\tfrac{1-\sqrt{5}}{2})^{n-2}\right) && \text{by inductive hypothesis} \\
& = (-1)^{n+1} + \tfrac{-1+\sqrt{5}}{2}(\tfrac{1+\sqrt{5}}{2})^{n+1} + \tfrac{-1-\sqrt{5}}{2}(\tfrac{1-\sqrt{5}}{2})^{n+1}
\end{aligned}$$