Problem Set 7 Solutions

Problem 7-1. **(a)** The answer is $\binom{52}{5}$.

(b) The answer is 4.

(c) The answer is $13 \cdot 48$.

(d) The answer is $13 \cdot 12 \cdot \binom{4}{2} \cdot \binom{4}{3}$.

(e) The answer is $4 \cdot \binom{13}{5}$.

(f) The answer is $9 \cdot 4^5$. (Some people assumed the Ace can appear at the beginning - Ace, two, three ...,- or at the end - ... Queen, King, Ace. You will get credit for this.)

(g) The answer is $13 \cdot 12 \cdot 11 \cdot \binom{4}{3} \cdot 4^2$.

(h) The answer is $\binom{13}{2} \cdot 11 \cdot \binom{4}{2}^2 \cdot 4$.

(i) The answer is $13 \cdot 12 \cdot 11 \cdot 10 \cdot \binom{4}{2} \cdot 4^3$.

(For parts g), h) and i) some people assumed you could have anything else as long as you had at least 3 of a kind, or at least 2 pairs, or at least 1 pair. You will get credit for this too.)

Problem 7-2. Any relation can be represented as an $n \times n$ matrix of 0's and 1's.

(a) 2^{n^2} since there are this many different $0-1$ matrices of order n.

(b) The matrix associated with a symmetric relation is symmetric. Since there are $2^{n+\binom{n}{2}}$ different symmetric matrices there are an equal number of symmetric relations.

(c) The matrix associated with a reflexive relation has 1's along the main diagonal. Hence the number of such matrices, and therefore reflexive relations, is 2^{n^2-n}.

(d) The matrix associated with a symmetric and reflexive relation is symmetric and has 1's along the main diagonal. Hence the number of such relations is $2^{\binom{n}{2}}$.

(e) The number of symmetric or reflexive relations = the number of reflexive relations + the number of symmetric relations − the number of reflexive and symmetric relations $= 2^{n^2-n} + 2^{n+\binom{n}{2}} - 2^{\binom{n}{2}}$.

(f) The number of equivalence relations is the Bell number $B_n = (1/e) \sum_{k=0}^{\infty} k^n/k!$.

Problem 7-3. **(a)** There are 26 letters in A,B,...,Z. If we use a maximum of 3 dots/dashes per character then we can represent only $2^1+2^2+2^3 = 13$ characters. Hence we need at least 4 and it is easy to see that $2^1 + 2^2 + 2^3 + 2^4 = 29 \geq 26$ and hence the maximum number of dots/dashes required by any character is 4.

(b) There are 62 letters in A,B,...,Z,a,b,...,z,0,1,...,9. Since $2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 61 < 62 < 125 = 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6$, the maximum number of dots/dashes required by any character is 6.

(c) Let a *valid k-sequence* be a sequence of dots/dashes that has no consecutive dots. Let $f(k)$ denote the number of valid k-sequences. If the first character in a valid k-sequence is a dash then the rest of the sequence forms a valid $k-1$ sequence. If the first character in a valid k-sequence is a dot then the second character must be a dash and the rest of the sequence forms a valid $k-2$ sequence. This gives us the (by now familiar) recurrence

$$f(k) = f(k-1) + f(k-2)$$

The characteristic equation $x^2 - x - 1$ has roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. We have boundary conditions $f(1) = 2$ and $f(2) = 2$. The solution to the recurrence is $f(k) = (2/\sqrt{5})((\frac{1+\sqrt{5}}{2})^k - (\frac{1-\sqrt{5}}{2})^k)$.

(d) Since the effective sending time of a dot is 2 seconds and the effective sending time of a dash is 4 seconds, we can divide everything by 2 and rescale back at the end. So for now let us think of the effective sending time of a dot as 1 second and that of a dash as 2 seconds. Let $g(k)$ denote the number of sequences with sending times at most k seconds. If the first symbol is a dot then the rest of the sequence must have sending time at most $k-1$ seconds. If the first symbol is a dash then the rest of the sequence must have sending time at most $k-2$ seconds. This gives us the (by now (uncomfortably) familiar) recurrence

$$g(k) = g(k-1) + g(k-2)$$

The characteristic equation $x^2 - x - 1$ has roots $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. We have boundary conditions $g(1) = 1$ and $g(2) = 3$. The solution to the recurrence is $((\frac{1+\sqrt{5}}{2})^k + (\frac{1-\sqrt{5}}{2})^k)$. Rescaling back we find the correct solution is $((\frac{1+\sqrt{5}}{2})^{\lfloor k/2 \rfloor} + (\frac{1-\sqrt{5}}{2})^{\lfloor k/2 \rfloor})$.

(In the problem set we asked for the number of sequences having sending times of less than k seconds. The above solution calculates the number of sequences having sending times of at most k seconds. Both interpretations will get credit.)

Problem 7-4. **(a)** Consider the number of ways of arranging r red and $n+1$ black balls in a line. This is equal to $\binom{n+r+1}{r}$. Now consider counting the number of ways based on the number k of red balls after the first black ball. There are $\binom{n+k}{k}$ such ways for all $k = 0, 1, \ldots, r$. Hence we are done.

(b) The proof is by induction on r. Let $P(r)$ be the statement $\sum_{k=0}^{r}\binom{n+k}{k} = \binom{n+r+1}{r}$.

Base Case. $P(0)$ is true since $\sum_{k=0}^{0}\binom{n+k}{k} = 1\binom{n+0+1}{0}$.

Inductive step. Assume $P(r)$ is true. Then

$$\begin{aligned}
\sum_{k=0}^{r+1}\binom{n+k}{k} &= \sum_{k=0}^{r}\binom{n+k}{k} + \binom{n+r+1}{r+1} \\
&= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} && \text{by inductive hypothesis} \\
&= \binom{n+r+2}{r+1} && \text{by Pascal's identity}
\end{aligned}$$

Hence $P(r+1)$ is true. So, $P(r)$ is true for all r and we are done.

Problem 7-5. **(a)** $\binom{100}{55}$

(b) $\binom{17}{8}3^8 2^9$

(c) x^k has a nonzero coefficient iff $-100 \le k \le 200$ and $k = 2 \bmod 3$. In such a case the coefficient is $\binom{100}{(100+k)/3}(-1)^{(200-k)/3}$

Problem 7-6. **(a)** Consider the number of ways of picking a team of size k from a group of n people and designating one of them captain. This can be done in $k\binom{n}{k}$ ways, for all $k = 0, 1, \ldots, n$. But the total number of ways of doing this can also be counted differently. First select the captain - n ways - then select an arbitrary subset of the rest - 2^{n-1} ways. Hence the result follows.

(b) Differentiating

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$$

we get

$$n(1+x)^{n-1} = \sum_{k=1}^{n}k\binom{n}{k}x^{k-1}$$

and evaluating it at $x = 1$ we get the desired result.

.