Massachusetts Institute of Technology
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Handout 37
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Notes 14

1 Binomial Coefficients

There are infinitely many interesting binomial coefficient identities. They can't all be learned, but a few standard tools let you rederive them as needed. One approach is algebraic, and usually works by brute force. A cleverer method of *combinatorial* proof builds a thought experiment that avoids all the math by using the semantics of the binomial coefficients.

$\binom{n}{0} = 1$. $\binom{n}{1} = n$.

Binomial coefficients are *unimodal:* $\binom{n}{r}$ increases with r until $r = n/2$, then decreases.

$$\binom{n}{r} / \binom{n}{r-1} = (n - r + 1)/r$$

Coefficients form a bell curve: suppose each student is equally likely to get each of n questions during the semester right or wrong. Then the bell curve describes how many ways the student can get K questions right, namely $\binom{n}{k}$.

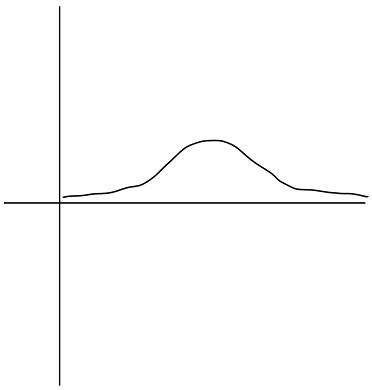


Figure 1: Bell Curve.

Theorem 1.1 $\binom{n}{r} = \binom{n}{n-r}$

Proof. (Algebraic)

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}.$$

■

Proof. (Combinatorial) Choosing r objects from n is the same as "unchoosing" the remaining $n-r$ objects. Thus there is a bijection between the sets of size r and those of size $n-r$. Hence, there is the same number of each. ■

Theorem 1.2 $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$

Proof. (Algebraic)

$$\begin{aligned} \binom{n}{r} &= \frac{n!}{r!(n-r)!} \\ &= \frac{n}{r}\frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{n}{r}\binom{n-1}{r-1} \end{aligned}$$

■

Proof. (Combinatorial) Consider choosing a committee and a leader of size r from among n guys. One way is to pick the leader first (n ways) and then pick the other $r-1$ committee members from the remainder ($\binom{n-1}{r-1}$ ways). The other is to pick the r committee members ($\binom{n}{r}$ ways) and then pick a leader from among them (r ways). Thus

$$n\binom{n-1}{r-1} = r\binom{n}{r}.$$

Now divide both sides by r. ■

Theorem 1.3 (Pascal)

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Proof. (Algebraic)

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!}$$

$$\begin{aligned} &= (n-r)\frac{(n-1)!}{r!(n-r)!} + r\frac{(n-1)!}{r!(n-r)!} \\ &= n\frac{(n-1)!}{r!(n-r)!} \\ &= \binom{n}{r} \end{aligned}$$

■

Proof. (Combinatorial) Let $S = \{1, \ldots, n\}$.

- $A = \{X \subseteq S \mid \|X\| = r\}$ (r-combs of S)
- $B = \{X \subseteq S \mid \|X\| = r \wedge 1 \in X\}$
- $C = \{X \subseteq S \mid \|X\| = r \wedge 1 \notin X\}$

We know that $A = B \cup C$ and B, C are disjoint.

- $\|B\| = \binom{n-1}{r-1}$
- $\|C\| = \binom{n-1}{r}$

Rule for sum says $\binom{n}{r} = \|A\| = \|B\| + \|C\|$. ■

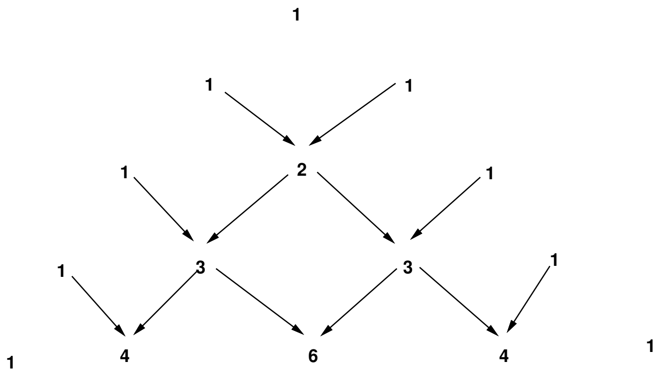


Figure 2: Pascal's triangle.

What happens if I add the coefficients in one row?

Theorem 1.4

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof. (induction) Let $P(n)$ be

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Base case. If $n = 1$ then

$$\sum_{k=0}^{1} \binom{1}{k} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1.$$

Hence $P(1)$ is true.
Inductive step. Assume $P(n-1)$ is true. Then

$$\begin{aligned}
\sum_{k=0}^{n+1} \binom{n+1}{k} &= \tbinom{n+1}{0} + \textstyle\sum_{k=1}^{n}(\tbinom{n}{k} + \tbinom{n}{k-1}) + \tbinom{n+1}{n+1} && \text{by Pascal's identity} \\
&= \tbinom{n}{0} + \textstyle\sum_{k=1}^{n-1} 2(\tbinom{n}{k}) + \tbinom{n}{n} \\
&= 2\textstyle\sum_{k=0}^{n} \tbinom{n}{k} \\
&= 2^{n+1} && \text{by inductive hypothesis.}
\end{aligned}$$

Hence, $P(n+1)$ is true. So $P(n)$ is true for all n. ∎

Proof. (combinatorial) Consider the collection of all possible subsets of an n element universe. There are 2^n such subsets. Now count the collection based on the sizes of the subsets. There are $\binom{n}{k}$ subsets of size k, for $k = 0, 1, \ldots, n$. Hence the theorem follows. ∎

Theorem 1.5 (Vandermonde)

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof. (Induction using Pascal's identity) Let $P(n)$ be

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Base Case. If $n = 0$ then

$$\binom{m+0}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{0}{k} = \binom{m}{r}$$

Hence $P(0)$ is true.
Inductive Step. Assume $P(n)$ is true. Then

$$\begin{aligned}
\binom{m+n+1}{r} &= \binom{m+n}{r} + \binom{m+n}{r-1} && \text{by Pascal's identity} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n}{k-1} && \text{by inductive hypothesis} \\
&= \binom{m}{r}\binom{n}{0} + \sum_{k=1}^{r} \binom{m}{r-k}(\binom{n}{k} + \binom{n}{k-1}) \\
&= \binom{m}{r}\binom{n+1}{0} + \sum_{k=1}^{r} \binom{m}{r-k}\binom{n+1}{k} && \text{by Pascal's identity} \\
&= \sum_{k=0}^{r} \binom{m}{r-k}\binom{n+1}{k}
\end{aligned}$$

Hence, $P(n+1)$ is true. So $P(n)$ is true for all n. ■

Proof. (Combinatorial) Suppose there are m red balls and n blue balls. Consider all possible ways to choose r balls. There are $\binom{m+n}{r}$ such ways. Now count the number of ways based on how many of the balls are blue.There are $\binom{m}{r-k}\binom{n}{k}$ ways to choose r balls such that k of them are blue, for $k = 0, 1, \ldots, r$. Hence the theorem follows. ■

2 The Binomial Theorem

Theorem 2.1 *Let $n \in \mathbb{N}$. Then for all x and y*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Example 1 $(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$

Proof. (Algebraic) By induction (HW) ■

Proof. (Combinatorial)

$$(x+y)(x+y)(x+y)(x+y) \;\; n = 4 \text{ binomials}$$

Each $x^k y^{n-k}$ term selects k binomials from the n to have x, rest y. $\binom{n}{k}$ ways of choosing binomials. ■

2.1 Identities via Binomial Theorem

Binomial theorem gives slick proofs of previous theorems. e.g.:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k} &= 2^n.\ \text{Plug in } x=1, y=1.(*)\\ \sum_{k=0}^{n}(-1)^k\binom{n}{k} &= 0.\ \text{Plug in } x=-1, y=1.(**)\end{aligned}$$

Combinatorial interpretation: number of ways of selecting an even number of elements from a set of n equals the number of ways of selecting an odd number.

$$\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n}{2k} = 2^{n-1}$$

$$\text{LHS} = \frac{(*)+(**)}{2} = \#\ \text{ways to pick even} = 2^{n-1}$$

Pascal's identity:

$$(x+y)^n = x(x+y)^{n-1} + y(x+y)^{n-1}$$

Vandermonde:

$$(1+t)^{m+n} = (1+t)^m(1+t)^n$$

3 General Binomial Theorem

Definition 3.1 General binomial coefficient $\binom{\alpha}{k}$ *for* $\alpha \in \mathbb{R}$ *and* $k \in \mathbb{N}$*:*

$$\begin{aligned}\binom{\alpha}{k} &= \tfrac{1}{k!}\textstyle\prod_{j=0}^{k-1}(\alpha-j)\\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\\ \binom{\alpha}{0} &= 1\end{aligned}$$

Theorem 3.2 *Let* $\alpha \in \mathbb{R}$ *and* $\left|\frac{x}{y}\right| < 1$*. Then*

$$(x+y)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^{\alpha-k}y^k$$

Proof. Calculus. ■

Example 2 *Approximate* $\sqrt{2}$

$$\sqrt{2} = (x+y)^{\frac{1}{2}},\ where\ x = 1, y = 1,\ but\ \left|\frac{x}{y}\right| \not< 1$$

$$\sqrt{2} = 2\frac{\sqrt{2}}{2} = 2\sqrt{\frac{1}{2}} = 2(1-\frac{1}{2})^{\frac{1}{2}} = 2(x+y)^{\frac{1}{2}},\ where\ x = 1, y = -\frac{1}{2}$$

$$(1-\frac{1}{2})^{\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(-\frac{1}{2}\right)^k$$

$$\binom{\frac{1}{2}}{0} = 1, \binom{\frac{1}{2}}{1} = \frac{1}{2}, \binom{\frac{1}{2}}{2} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}, \binom{\frac{1}{2}}{3} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{1}{16}, etc.$$

$$\begin{aligned} \sqrt{2} &= 2\left(1 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right) + \left(-\tfrac{1}{8}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{1}{16}\right)\left(-\tfrac{1}{8}\right) + \cdots\right) \\ &= 2\left(1 - \tfrac{1}{4} - \tfrac{1}{32} - \tfrac{1}{128} - \cdots\right) \\ &\approx 2(.7109375) = 1.421875 \end{aligned}$$

4 Principle of Inclusion and Exclusion

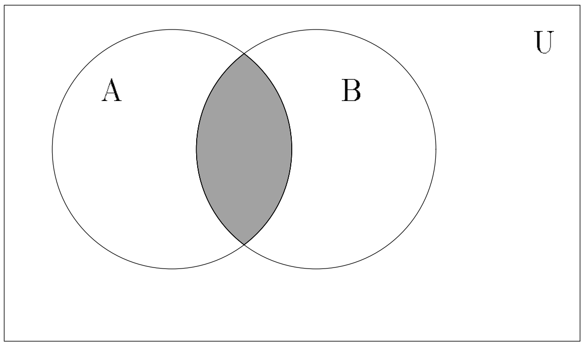
Problem: How many numbers form $\{1, \cdots, 100\}$ are divisible by 3 or 5?

By 3: $\lfloor\frac{100}{3}\rfloor = 33$
By 3: $\lfloor\frac{100}{5}\rfloor = 20$

3 or 5: Rule of Sum $\Rightarrow 33 + 20 = 53$. WRONG! Thus counts number divisible by 15 twice; must subtract out.

By 15: $\lfloor\frac{100}{15}\rfloor = 6$

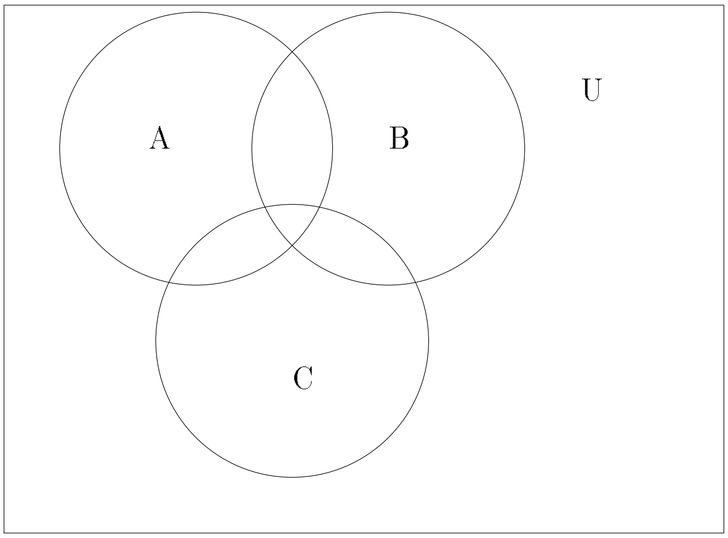
3 or 5: 33+20-6=47



Venn diagram:

$|A \cup B| = |A| + |B| - |A \cap B|$

3 sets:



$|A \cup B| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

In general, for m sets,

$$|\cup_i A_i| = \sum_i |A_i| - \sum_{i \neq j} |A_i \cap A_j| + \sum_{i,j,k \text{ distinct}} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{m+1} |\cap_i A_i|$$

Prove by induction on the number of sets.

Often, see inclusion/exclusion stated in terms of what is **not** in sets $A_i \subseteq U$

$$\left|\cap_i \overline{A_i}\right| = |U| - \sum_i |A_i| + \sum_{i \neq j} |A_i \cap A_j| - \sum_{i,j,k \text{distinct}} |A_i \cap A_j \cap A_k| - \cdots + (-1)^m |\cap_i A_i|$$

For m sets, 1st term has 1 set, 2nd term has m sets, 3rd term has $\binom{m}{2}$ sets, $\cdots$, kth term has $\binom{m}{k}$ sets.

Example 3 How many primes less than 100? 100 minus numbers divisible by $2, 3, 5$ or 7 plus these 4 numbers. Let

$$P_2 = \text{numbers divisible by } i$$

note $\|P_i\| = \lfloor 100/i \rfloor$.

Then quantity is:

$$99 \quad - \quad \lfloor\frac{100}{2}\rfloor - \lfloor\frac{100}{3}\rfloor - \lfloor\frac{100}{5}\rfloor - \lfloor\frac{100}{7}\rfloor$$

$$
\begin{aligned}
+ \quad & \lfloor\frac{100}{2\cdot 3}\rfloor + \lfloor\frac{100}{2\cdot 5}\rfloor + \lfloor\frac{100}{2\cdot 7}\rfloor + \lfloor\frac{100}{3\cdot 5}\rfloor + \lfloor\frac{100}{3\cdot 7}\rfloor + \lfloor\frac{100}{5\cdot 7}\rfloor \\
- \quad & \lfloor\frac{100}{2\cdot 3\cdot 5}\rfloor - \lfloor\frac{100}{2\cdot 3\cdot 7}\rfloor - \lfloor\frac{100}{2\cdot 5\cdot 7}\rfloor - \lfloor\frac{100}{3\cdot 5\cdot 7}\rfloor \\
+ \quad & \lfloor\frac{100}{2\cdot 3\cdot 5\cdot 7}\rfloor
\end{aligned}
$$

which is

$$99 - 50 - 33 - 20 - 14 + 16 + 10 + 7 + 6 + 4 + 2 - 3 - 2 - 1 - 0 + 0 = 21$$

so there are 25 primes.