Problem Set 6 Solutions

Problem 6-1. Present value of what the college loan officer offers

$$
\begin{aligned}
&= 25000 + 25000/1.05 + 25000/(1.05)^2 + 25000/(1.05)^3 \\
&= 25000(1 - \frac{1}{(1.05)^4})/(1 - \frac{1}{1.05}) \\
&= 25000 * 3.72 \\
&= \$93,081.20
\end{aligned}
$$

Present value of what you must pay in return

$$
\begin{aligned}
&= 15000/(1.05)^4 + 15000/(1.05)^5 + \ldots + 15000/(1.05)^{18} \\
&= (15000/(1.05)^4)(1 - \frac{1}{(1.05)^{15}})/(1 - \frac{1}{1.05}) \\
&= 15000 * 8.97 \\
&= \$134,495.08
\end{aligned}
$$

So this is not quite a fair deal for you. Of course, in reality the loan officer must charge a premium for default risk.

Problem 6-2. We first come up with a formula for $\sum_{i=1}^{n} i^2 x^i$ and then prove it using induction. From class notes we know that

$$\sum_{i=1}^{n} i x^i = \frac{x - (n+1)x^{n+1} + n x^{n+2}}{(1-x)^2}.$$

Differentiating (with respect to x) the above expression and multiplying with x we find that

$$\sum_{i=1}^{n} i^2 x^i = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

Proof by induction. Let $P(n)$ be the statement that

$$\sum_{i=1}^{n} i^2 x^i = \frac{x + x^2 - (n+1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2 x^{n+3}}{(1-x)^3}.$$

Base Case. $P(1)$ is true since

$$\sum_{i=1}^{1} i^2x^i = x = \frac{x + x^2 - 4x^2 + 3x^3 - x^4}{(1-x)^3}.$$

Inductive Step. Assume $P(n)$ is true. Then

$$\begin{aligned}
\sum_{i=1}^{n} i^2x^i + (n+1)^2x^{n+1} &= \\
\sum_{i=1}^{n+1} i^2x^i &= \frac{x + x^2 - (n+1)^2x^{n+1} + (2n^2+2n-1)x^{n+2} - n^2x^{n+3}}{(1-x)^3} + (n+1)^2x^{n+1} \\
&= \frac{x + x^2 - (n+2)^2x^{n+2} + (2(n+1)^2 + 2(n+1) - 1)x^{n+3} - (n+1)^2x^{n+4}}{(1-x)^3}
\end{aligned}$$

Thus $P(n+1)$ is true. Hence, $P(n)$ is true for all n.

Problem 6-3. We use the integral method for bounding S_n.

$$\int_1^{n+1} \sqrt{x}dx \geq S_n \geq \int_0^n \sqrt{x}dx.$$

i.e.

$$\frac{(2/3)((n+1)^{3/2} - 1)}{n^{3/2}} \geq \frac{S_n}{n^{3/2}} \geq \frac{(2/3)n^{3/2}}{n^{3/2}}$$

Taking limits as $n \to \infty$ we find that

$$2/3 \geq \lim_{n\to\infty} \frac{S_n}{n^{3/2}} \geq 2/3$$

i.e. $\lim_{n\to\infty} \frac{S_n}{n^{3/2}} = 2/3$.

Problem 6-4.

$$\begin{aligned}
T(n) &= aT(n/2) + bn \\
&= a^2T(n/2^2) + abn/2 + bn \\
&= \ldots \\
&= a^iT(n/2^i) + bna^{i-1}/2^{i-1} + bna^{i-2}/2^{i-2} + \ldots + bn \\
&= \ldots \\
&= a^{\log_2 n}T(1) + bn(1 + a/2 + (a/2)^2 + \ldots + (a/2)^{\log_2 n - 1})
\end{aligned}$$

Case 1. $a < 2$. Then it is easy to see from the above that

$$\begin{aligned} T(n) &= a^{\log_2 n} T(1) + bn(1 + a/2 + (a/2)^2 + \ldots + (a/2)^{\log_2 n - 1}) \\ &= n^{\log_2 a} T(1) + n\Theta(1) \\ &= \Theta(n) \end{aligned}$$

Case 2. $a = 2$. Then it is easy to see from the above that

$$\begin{aligned} T(n) &= a^{\log_2 n} T(1) + bn(1 + a/2 + (a/2)^2 + \ldots + (a/2)^{\log_2 n - 1}) \\ &= nT(1) + bn\log_2 n \\ &= \Theta(n \log_2 n) \end{aligned}$$

Case 3. $a > 2$. Then it is easy to see from the above that

$$\begin{aligned} T(n) &= a^{\log_2 n} T(1) + bn(1 + a/2 + (a/2)^2 + \ldots + (a/2)^{\log_2 n - 1}) \\ &= n^{\log_2 a} T(1) + bn((a/2)^{\log_2 n} - 1)/((a/2) - 1) \\ &= \Theta(n^{\log_2 a}) \end{aligned}$$

Problem 6-5. **(a)** Using iteration we find $T(n) = 3^{\log_2 n} T(1) + n^2(1 + 3/2^2 + (3/2^2)^2 + \ldots + (3/2^2)^{\log_2 n - 1}) = n^{\log_2 3} T(1) + n^2\Theta(1) = \Theta(n^2)$.

(b) It is clear that $T(n) = \Omega(n)$. We show by induction that $T(n) = O(n)$ thus proving that $T(n) = \Theta(n)$. To do this, we show that $\exists$ a constant c such that $\forall n, T(n) \leq cn$. We pick $c = 2$. Let $P(n)$ be the statement that $T(n) \leq cn$.
Base Case. $T(1) = 1 \leq 2 \cdot 1$. Hence $P(1)$ is true.
Induction Step. Assume $P(k)$ is true for all $k \leq n$. Then

$$\begin{aligned} T(n+1) &= 2T(\lfloor \frac{n+1}{5} \rfloor) + (n+1) \\ &\leq 2 \cdot 2 \cdot (\lfloor \frac{n+1}{5} \rfloor) + (n+1) \\ &\leq 4n/5 + (n+1) \\ &\leq 2(n+1) \end{aligned}$$

(We made use of the inductive hypothesis to infer that $T(\lfloor \frac{n+1}{5} \rfloor) \leq 2 \cdot (\lfloor \frac{n+1}{5} \rfloor)$.)
Thus we have proved $P(n+1)$. Hence $P(n)$ is true for all n.

(c) Using iteration we find that $T(n) = 4^{\log_4 n} T(1) + n \log_4 n = \Theta(\log n)$.

(d) It is clear that $T(n) = \Omega(n)$. We show by induction that $T(n) = O(n)$ thus proving that $T(n) = \Theta(n)$. To do this, we show that $\exists$ a constant c such that $\forall n, T(n) \leq cn$. We pick $c = 12$. Let $P(n)$ be the statement that $T(n) \leq cn$.

Base Case. $T(1) = 1 \leq 12 \cdot 1$. Hence $P(1)$ is true.
Induction Step. Assume $P(k)$ is true for all $k \leq n$. Then

$$\begin{aligned} T(n+1) &= T(\lfloor\frac{n+1}{2}\rfloor) + T(\lfloor\frac{n+1}{3}\rfloor) + (n+1) \\ &\leq 12 \cdot (\lfloor\frac{n+1}{2}\rfloor) + 12 \cdot (\lfloor\frac{n+1}{3}\rfloor) + (n+1) \\ &\leq 6(n+1) + 4(n+1) + (n+1) \\ &\leq 12(n+1) \end{aligned}$$

(We made use of the inductive hypothesis to infer that $T(\lfloor\frac{n+1}{2}\rfloor) \leq 12 \cdot (\lfloor\frac{n+1}{2}\rfloor)$ and $T(\lfloor\frac{n+1}{3}\rfloor) \leq 12 \cdot (\lfloor\frac{n+1}{3}\rfloor)$.) Thus we have proved $P(n+1)$. Hence $P(n)$ is true for all n.

Problem 6-6. **(a)**

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} x^i y^{i+2j} &= (\sum_{i=1}^{\infty}(xy)^i)(\sum_{j=1}^{\infty}(y^2)^j) \\ &= (xy/(1-xy)) \cdot (y^2/(1-y^2)) \\ &= (xy^3)/((1-xy)(1-y^2)) \end{aligned}$$

(b)

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ijx^i y^j &= (\sum_{i=1}^{\infty} ix^i)(\sum_{j=1}^{\infty} jy^j) \\ &= (x/(1-x)^2) \cdot (y/(1-y)^2) \\ &= (xy)/((1-x)(1-y))^2 \end{aligned}$$

Problem 6-7. **(a)**

$$f(i) = (1/2)(f(i-1) + f(i-2))$$

This is a homogenous linear recurrence. We solve the characteristic equation, $2x^2 - x - 1 = 0$, to get the roots 1 and $-1/2$. Hence the general solution is of the form $f(i) = A + B(-1/2)^i$. Using the boundary conditions $f(0) = 100$ and $f(1) = 300$ we find that the general solution is $f(i) = 700/3 - (400/3)(-1/2)^i$.

(b)

$$g(i) = (1/3)(g(i-1) + g(i-2) + g(i-3))$$

This is a homogenous linear recurrence. We solve the characteristic equation, $3x^3 - x^2 - x - 1 = 0$, to get the roots 1, $\frac{-1-\sqrt{-2}}{3}$ and $\frac{-1+\sqrt{-2}}{3}$. Hence the general solution is of the form $g(i) = A + B(\frac{-1-\sqrt{-2}}{3})^i + C(\frac{-1+\sqrt{-2}}{3})^i$. Using the boundary conditions $g(0) = 100$, $g(1) = 300$ and $g(2) = 200$ we find that the general solution is $g(i) = 650/3 + (-25/3)(7 - 4\sqrt{-2})(\frac{-1-\sqrt{-2}}{3})^i + (-25/3)(7 + 4\sqrt{-2})(\frac{-1+\sqrt{-2}}{3})^i$.

(c)

$$h(i) = (1/2)(h(i-1) + h(i-2)) + i$$

For the homogenous part we get the same characteristic equation as in part a). Hence the homogenous solution is of the form $C + D(-1/2)^i$. We guess a particular solution of the form $Ai^2 + Bi$, plug into the recurrence and equate coefficients of like powers on both sides, to get $A = 1/3$ and $B = 5/9$. Now we use the boundary conditions $h(0) = 100$ and $h(1) = 300$ to find that the general solution is $h(i) = 6284/27 + (-3584/27)(-1/2)^i + (1/3)i^2 + (5/9)i$.