Problem Set 5 Solutions

Problem 5-1. Let P_n denote a directed path on n vertices. That is, it is a graph consisting of vertices $v_1, v_2, \ldots, v_n$, and directed edges $(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)$.

Theorem 0.1 *The Hasse diagram of a totally ordered set on n vertices is equal to P_n.*

Proof. We prove this by induction on n. Let $Q(n)$ denote the statement, "The Hasse diagram of a totally ordered set on n vertices is equal to P_n."

Base Case: $n = 1$. Here the Hasse diagram is a single vertex with no edges, which is precisely P_1.

Inductive Step. We assume $Q(n)$, and show $Q(n+1)$. Let X be a totally ordered set on $n+1$ elements. We know that every poset contains a minimal element; let x be a minimal element of X. Let $X' = X - \{x\}$.

We claim that X' is a totally ordered set on n elements. To see this, note that X' does have exactly n elements, and any two elements of X' are comparable (since they are comparable in the larger set X). Thus, by the induction hypothesis, the Hasse diagram of X' is the graph P_n.

Since the Hasse digram of X' is P_n, X' contains a unique minimal element (the element corresponding to the vertex v_1). Let x' denote this element.

To construct the Hasse diagram of X, we need to add to the Hasse diagram for X' a vertex representing x, with edges from x representing all its comparability relations. Observe that since X is totally ordered, $x \preceq y$ for all $y \in X$. Additionally, since X' is totally ordered, $x' \preceq y'$ for all $y' \in X'$. Thus we need only introduce a single new edge, from the vertex representing x to the vertex representing x'. All the remaining comparability relationships involving x will be implied by transitivity.

So the Hasse digram for X consists of a directed path on $n+1$ vertices, which is precisely P_{n+1}. This completes the proof. ∎

Problem 5-2. **(a)** We prove this by contradiction. Suppose that the set of tasks can be completed in $T < D$ time units. Let C be a chain in the poset of maximum length (i.e. D) ; then by the pigeonhole principle, some two tasks in C must be executed in the same time unit. But since these two tasks are related in the poset, they must be done in different time units — a contradiction.

(b) We prove this too by contradiction. Suppose that the set of tasks can be completed in $T < N/P$ time units. Then by the pigeonhole principle, there is some time unit in which more than P jobs are being performed. As we have only P processors, we can apply the pigeonhole principle again to conclude that in this particular time unit, there is some processor that is performing more than one task. But this contradicts the assumption that each processor can only perform one task per time unit.

(c) This part we prove by an actual construction of a schedule for the processors. Since the poset has maximum chain length D, we know from class that it can be partitioned into antichains $A_1, A_2, \ldots, A_D$, with the following property:

(*) For any $a_i \in A_i$: $b \preceq a_i$ implies $b \in A_1 \cup \cdots \cup A_{i-1}$.

Suppose that for $i = 1, \ldots, D$, $|A_i| = N_i$. Our schedule for the processors is as follows.

- For $i = 1, 2, \ldots, D$
  - Perform the tasks in antichain A_i, P tasks per time unit.

First we claim this a valid schedule. This is by property (*) — by the time that a task a is executed, all the tasks which a depends on have been executed (in previous antichains).

Let's analyze how long this schedule takes. To perform the tasks in antichain A_i, since they can be done P at a time, takes $\lceil N_i/P \rceil$ time units. (The "ceiling" notation is because N_i may not divide P evenly. So for example, if we have $N_i = 5$ and $P = 4$, it will take us 2 time units to perform the tasks in antichain A_i.) Since

$$\lceil N_i/P \rceil \leq 1 + N_i/P,$$

the total time to perform the tasks in A_i is at most $1 + N_i/P$.

So the total time is at most

$$\begin{aligned} \sum_{i=1}^{D}(1 + N_i/P) &= \sum_{i=1}^{D} 1 + (1/P)\sum_{i=1}^{D} N_i \\ &= D + N/P \end{aligned}$$

Problem 5-3. **(a)** By a circle, let's say that we mean the "filled-in" interior of a circle in the plane.

Let S be an arbitrary set of N circles in the plane, labeled $C_1, \ldots, C_N$. Put the following relation $\preceq$ on circles: we say that $C_i \preceq C_j$ if and only if C_i is contained in C_j. (That is, if and only if $C_i \subset C_j$.) We claim this is a partial order, which we verify as follows.

- Reflexive: A circle is contained in itself, by the reflexivity of the subset relation.

- Anti-Symmetric: If C_i is contained in C_j, and C_j is contained in C_i, then $C_i = C_j$: this follows from the definition of set equality.
- Transitive: If C_i is contained in C_j, and C_j is contained in C_k, then C_i is contained in C_k: this follows from the transitivity of the subset relation.

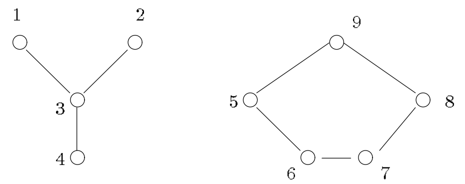
Now, in this poset $(S, \preceq)$, a chain corresponds to a set of circles so that for any pair of them, one is contained in the other. This is a *good* set. An antichain corresponds to a set of circles none of which is contained in any other. This is a *bad set*. Since any poset contains either a chain or antichain of size at least $\sqrt{N}$, this proves the theorem.

(b) Let S be a set of N natural numbers, labeled $x_1, \ldots, x_N$. Put the following relation $\preceq$ on circles: we say that $x_i \preceq x_j$ if and only if $x_i | x_j$. We claim this is a partial order, which we verify as follows.

- Reflexive: Since $x_i = 1(x_i)$, $x_i | x_i$ and so $x_i \preceq x_i$.
- Anti-symmetric. Suppose $x_i | x_j$ and $x_j | x_i$. Then $x_j = ax_i$ and $x_i = bx_j$. So $x_i = abx_i$, which implies that $ab = 1$. Since a and b are positive natural numbers, this means that $a = b = 1$, and hence that $x_i = x_j$.
- Transitive: Suppose $x_i | x_j$ and $x_j | x_k$. So $x_j = ax_i$ and $x_k = bx_j$, so $x_i = abx_k$, and we have $x_i | x_k$.

Now, in this poset $(S, \preceq)$, a chain corresponds to a set of numbers so that for any pair of them, one divides the other. This is a *good* set. An antichain corresponds to a set of numbers none of which divides any other. This is a *bad set*. Since any poset contains either a chain or antichain of size at least $\sqrt{N}$, this proves the theorem.

Problem 5-4. **(a)** The graph has two connected components.



(b) Nodes $1, 2, 4$ have degree 1, nodes $5, 6, 7, 8, 9$ have degree 2, and node 3 has degree 3.

(c)

$$\begin{pmatrix} 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 3 & 1 & 1 & 3 & 0 \end{pmatrix}$$

This matrix is the cube of the adjacency matrix.

Problem 5-5. By a *leaf* in a graph, we mean a vertex of degree 1. We first prove that every tree on at least 2 nodes has a leaf. (This was actually proved in recitation.) Let T be a tree, and let P be a path in T (with no repeated vertices) of maximum length. Suppose that $P = \{v_1, v_2 \ldots, v_k\}$ in this order. We claim that v_1 is a leaf. For if not, then there is an edge (v, v_1) in T, where $v \neq v_2$. Now if $v \notin P$, we can attach v to the end of P to obtain a longer path — a contradiction. But if $v \in P$, then since $v \neq v_2$, this will give a cycle in T — also a contradiction.

Given this fact, we prove the theorem by induction on n.

Base case: $n = 1$. We can color the single node anything we want, thereby using at most two colors.

Inductive step. Suppose that we know that every tree on n nodes is bipartite, for some $n \geq 1$. Let T be a tree on $n+1$ nodes; we need to show how to two-color it. Since $n+1 \geq 2$, we know that T has a leaf v. Let w be the unique neighbor of v. Let $T' = T - \{v\}$. Note that T' is a tree, since it is still connected and has no cycles. Thus, by the induction hypothesis, we can two-color it.

Now we define the following two-coloring of T. If u is a vertex of T not equal to v, give u the color that it had in our two-coloring of T'. Then give v whichever color we didn't give to its neighbor w. As this is a valid two-coloring of T, this completes the inductive step and the proof.

Problem 5-6. The "theorem" is in fact incorrect; consider the graph we used in question 4 as a counterexample. This problem was supposed to show an example of the dangers of "build-up" error.

The base case is fine — $P(2)$ really is true.

The trouble with the proof is our "build-up" step in the induction, where we form an $(n+1)$-node graph by attaching a vertex to an n-node graph of minimum degree at least one. This does *not* produce all possible $(n+1)$-node graphs of minimum degree at least one, and so the set of graphs for which we end up proving the theorem is *not* the set of all graphs of minimum degree at least one.

For example, the following graph is not produced by this "building-up" process: vertices A, B, C, D, and two edges (A, B) and (C, D). (To see this, ask yourself how you could get this from a 3-node graph of minimum degree one — the answer is that you can't.)