Problem Set 2 Solutions

Problem 2-1. Let $P(n)$ be

$$\text{“}\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6.\text{”}$$

BASIS STEP: $P(1)$ is true since

$$\sum_{j=1}^{1} j^2 = 1 = 1(1+1)(2\cdot 1+1)/6$$

.
INDUCTIVE STEP: Assume that

$$\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$$

. Then

$$\begin{aligned}\sum_{j=1}^{n+1} j^2 &= (\sum_{j=1}^{n} j^2) + (n+1)^2 \\ &= n(n+1)(2n+1)/6 + (n+1)^2 \\ &= (n+1)(2n^2+n+6n+6)/6 \\ &= (n+1)((n+1)+1)(2(n+1)+1)/6.\end{aligned}$$

This shows that $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.

Problem 2-2. Definition of multiplication:

- $0 \times n = 0$
- $m' \times n = (m \times n) + n$

We prove, by induction, that $m \times (n+p) = (m \times n) + (m \times p)$. Let $S = \{m | (\forall n \forall p) m \times (n+p) = (m \times n) + (m \times p)\}$. We prove $S = N$.

- $0 \in S$, since $0 \times (n+p) = (0 \times n) + (0 \times p) = 0 + 0 = 0$.

- Suppose $m \in S$ so that $m \times (n+p) = (m \times n) + (m \times p)$. Then

$$\begin{aligned} m' \times (n+p) &= m \times (n+p) + (n+p) && \text{defn of } \times \\ &= m \times n + m \times p + n + p && \text{inductive hypothesis} \\ &= (m \times n + n) + (m \times p + p) && \text{commutativity of } + \\ &= m' \times n + m' \times p && \text{defn of } \times. \end{aligned}$$

Therefore $S = N$ by induction axiom.

OR

Definition of multiplication:

- $m \times 0 = 0$
- $m \times n' = (m \times n) + m$

We prove, by induction, that $m \times (n+p) = (m \times n) + (m \times p)$. Let $S = \{p | (\forall m \forall n) m \times (n+p) = (m \times n) + (m \times p)\}$. We prove $S = N$.

- $0 \in S$, since $m \times (n+0) = (m \times n) + (m \times 0) = m \times n$.
- Suppose $p \in S$ so that $m \times (n+p) = (m \times n) + (m \times p)$. Then

$$\begin{aligned} m \times (n+p') &= m \times ((n+p)') && \text{defn of } + \\ &= m \times (n+p) + m && \text{defn of } \times \\ &= (m \times n) + ((m \times p) + m) && \text{inductive hypothesis} \\ &= (m \times n) + (m \times p') && \text{defn of } \times. \end{aligned}$$

Therefore $S = N$ by induction axiom.

Problem 2-3. Let $P(n)$ be "a full binary tree on n nodes has $(n+1)/2$ leaves."
BASIS STEP: $P(1)$ is true since a full binary tree on 1 node has $(1+1)/2 = 1$ leaf.
INDUCTIVE STEP: Assume $P(n)$ is true. Consider a full binary tree on $n+1$ nodes. Let there be k nodes in its left subtree. Then $k \leq n-1$ and its right subtree has $n-k$ nodes. By the inductive hypothesis, its left subtree has $(k+1)/2$ leaves and its right subtree has $(n-k+1)/2$ leaves. Hence the tree has $(k+1)/2 + (n-k+1)/2 = (n+1)/2$ leaves. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.

Problem 2-4. **Lemma 0.1** *In any sequence of n zeroes and ones that begins with a zero and ends with a one, there is a zero immediately preceding a one somewhere in the sequence.*

Proof. Move from the left end of the sequence to the right end. At some point we must move from a 0 to a 1 since we start at a 0 and end with a 1. Consider the first time we move from a 0 to a 1, at this point we have a 0 immediately preceding a 1. This completes the proof. ■

OR

Proof. We prove it by induction on the position on the length of the sequence. Let $P(n)$ be "if the sequence has length n then the sequence has a 0 immediately preceding a 1."
BASIS STEP: $P(2)$ is true since a sequence that has a 0 in position one and a 1 in position two has a 0 immediately preceding a 1.
INDUCTIVE STEP: Assume $P(n)$ is true. Consider a sequence of length $n+1$. If it has a 1 in position two, we are done. Otherwise it has a 0 in position two, in which, case consider the subsequence starting from position two and going to position $n+1$. This subsequence satisfies the preconditions of the lemma and has length n, hence by the inductive hypothesis, it and therefore the original sequence, contains a 0 immediately preceding a 1. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.

■

Having proved the lemma we move onto the main problem.

Let $P(n)$ be "every tournament with n teams has a chain."
BASIS STEP: $P(2)$ is true since a tournament with 2 teams has a winner and a loser and hence a chain.
INDUCTIVE STEP: Assume $P(n)$ is true. Consider a tournament with $n+1$ teams. Now split the $n+1$ teams arbitrarily into one team, called team $n+1$, and the remaining n teams. Note that the tournament contains a sub-tournament on these n teams which, by the inductive hypothesis must contain a chain. Let the teams be numbered $1, 2, \ldots, n$ according to their position in the chain. Now create a sequence of length n of 0s and 1s by considering the games between the team numbered $n+1$ and the rest. Put a 1 at position i in the sequence if team $n+1$ beat team i. Pad the sequence with a 0 at the beginning and a 1 at the end. By the above lemma there must be a position j at which there is a 0 immediately preceding a 1. It is clear that inserting team $n+1$ in the chain at position j generates a valid chain. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.

No, it does not mean the first team in the order always beat all the other teams. A simple counterexample can be got by considering a tournament with three teams A, B and C where A beats B, B beats C and C beats A.

Problem 2-5. **(a)** The inductive step does not work for $n = 0$ since a^{n-1} is undefined.

(b) The inductive hypothesis is invoked wrongly in the inductive step. r and s need not be squares, they could also be primes.

(c) The theorem is plainly false since there are only a finite number of thirteen word descriptions (assuming, of course that the English language has a finite number of words, a not unreasonable assumption) but there are an infinite number of positive integers. The ambiguity comes because of the meaning of the word *description*. If by description we mean a fixed (and computable) correspondence (from the ordered collection of thirteen words to a positive integer) then the sentence "the smallest number that cannot be described in less than fourteen English words" would either not *describe* a number or if it did then it would correspond to that number and would not carry its self-referential English meaning at the same time.

Problem 2-6. Let $P(n)$ be "F_{3n-1} is even."
BASIS STEP: $P(1)$ is true since $F_{3\cdot 1-1} = F_2 = 2$ is even.
INDUCTIVE STEP: Assume $P(n)$ is true. Then $F_{3(n+1)-1} = F_{3n+2} = F_{3n+1} + F_{3n}$, by definition. But $F_{3n+1} = F_{3n} + F_{3n-1}$. Hence $F_{3n+2} = 2F_{3n} + F_{3n-1}$. But F_{3n-1} is even by the inductive hypothesis, and hence so is F_{3n+2}. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.

OR

Let $P(n)$ be "F_{3n-1} is even, F_{3n-2} and F_{3n-3} are odd."
BASIS STEP: $P(1)$ is true since $F_{3\cdot 1-1} = F_2 = 2$ is even; and $F_{3\cdot 1-2} = F_1 = 1$ and $F_{3\cdot 1-3} = F_0 = 1$ are odd.
INDUCTIVE STEP: Assume $P(n)$ is true. Then $F_{3(n+1)-3} = F_{3n} = F_{3n-1} + F_{3n-2}$, by definition. But F_{3n-1} is even and F_{3n-2} odd, by the inductive hypothesis, and hence F_{3n} is odd. Similarly, $F_{3(n+1)-2}$ and $F_{3(n+1)-1}$ can be seen to be odd and even, respectively. Thus $P(n+1)$ is true. Hence $P(n)$ is true for all natural numbers n.