Problem Set 1 Solutions

Problem 1-1. **(a)** $r \Rightarrow p$.

(b) $p \bigwedge \neg q \bigwedge r$.

(c) $(p \bigwedge q) \Rightarrow r$.

(d) $r \Leftrightarrow (p \bigvee q)$.

Problem 1-2. $(p \bigwedge q \bigwedge \neg r) \bigvee (p \bigwedge \neg q \bigwedge r)(\neg p \bigwedge q \bigwedge r)$.

Problem 1-3. **(a)** $\exists x\ (P(x) \bigwedge Q(x))$.

(b) $\exists x\ (P(x) \bigwedge \neg Q(x))$.

(c) $\forall x\ (P(x) \bigvee Q(x))$.

(d) $\forall x\ \neg(P(x) \bigvee Q(x))$.

(e) $\forall x\ P(x) \Rightarrow \neg Q(x)$.

Problem 1-4. **(a)** $\forall x \exists y\ L(x, y)$.

(b) $\exists y \forall x\ L(x, y)$.

(c) $\forall x \exists y\ \neg L(x, y)$.

(d) $\exists x(\forall y\ L(y, x) \bigwedge \forall z((\forall w\ L(w, z)) \Rightarrow z = x))$.

(e) $\exists x \exists y(x \neq y \bigwedge L(Lynn, x) \bigwedge L(Lynn, y) \bigwedge \forall z(L(Lynn, z) \Rightarrow (z = x \bigvee z = y)))$.

(f) $\exists x \forall y(L(x, y) \Rightarrow x = y)$.

Problem 1-5. **(a)** $\forall x(P(x) \Rightarrow \neg Q(x))$.

(b) $\forall x(Q(x) \Rightarrow R(x))$.

(c) $\forall x(P(x) \Rightarrow \neg R(x))$.

(d) The conclusion does not follow. There could be other vain people besides ignorant ones and hence there may be vain professors.

Problem 1-6. $\forall x P(x) \bigvee \forall x Q(x) \Rightarrow \forall x \forall y (P(x) \bigvee Q(y))$. For $\forall x P(x) \bigvee \forall x Q(x)$ to be true it must be the case that $\forall x P(x)$ or $\forall x Q(x)$ is true. In either case $\forall x \forall y (P(x) \bigvee Q(y))$ becomes true.

$\forall x P(x) \bigvee \forall x Q(x) \Leftarrow \forall x \forall y (P(x) \bigvee Q(y))$. For $\forall x \forall y (P(x) \bigvee Q(y))$ to be true it must be the case that $\forall x P(x)$ or $\forall x Q(x)$ is true. Because if not, then there exists an x' and a y' such that $(P(x) \bigvee Q(y))$ is false.

Hence $\forall x P(x) \bigvee \forall x Q(x) \Leftrightarrow \forall x \forall y (P(x) \bigvee Q(y))$.

Problem 1-7. **(a)** Fallacy of affirming the conclusion.

(b) Fallacy of begging the question.

(c) Valid argument using modus tollens.

(d) Fallacy of denying the hypothesis.

Problem 1-8. Suppose that r is rational and i is irrational and $s = r + i$ is rational. Then $s + (-r) = i$ is rational, a contradiction.

Problem 1-9. There are four cases.

1. $x \geq 0$ and $y \geq 0$. Equality holds.
2. $x < 0$ and $y < 0$. Equality holds.
3. $x \geq 0$ and $y < 0$. If $x \geq -y$ then $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$ then $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$.
4. $x < 0$ and $y \geq 0$. Same as the previous case with the roles of x and y reversed.

Problem 1-10. Let the squares of the checkerboard be numbered with Cartesian coordinates so that the square in the lower lefthand corner gets the number $(1, 1)$. Then the square in the upper righthand corner gets numbered $(8, 8)$. Consider the the three squares $(1, 1), (5, 1)$ and $(3, 3)$. These three squares are pairwise a distance of 4 apart. Since two of these three squares must get the same color we see that there are two squares at distance exactly 4 apart that have the same color.

Problem 1-11. **Theorem 1**

$$\forall m, n, p \in \mathbb{N}, (m + p = n + p) \Rightarrow m = n$$

Proof. Fix m, n. Let $S = \{p \in \mathbb{N} \mid (m + p = n + p) \Rightarrow m = n\}$. We now prove $S = \mathbb{N}$ by induction.

- **Base case:** $m + 0 = n + 0 \Rightarrow m = n$ by defn of $+$.
- **Inductive step:** Suppose $p \in S$. Then, $(m + p = n + p) \Rightarrow m = n$.

$$\begin{aligned} m + p' = n + p' &\Rightarrow (m+p)' = (n+p)' \quad \text{(defn of +)} \\ &\Rightarrow m + p = n + p \quad \text{(Third Peano axiom)} \\ &\Rightarrow m = n \quad \text{(true for } p\text{)} \end{aligned}$$

Therefore, $S = \mathbb{N}$ by induction axiom. By universal generalization, $\forall m, n, p \in \mathbb{N}, (m+p = n + p) \Rightarrow m = n$. ■

Problem 1-12. The function

$$f(x) = \overbrace{x + x + \ldots + x}^{x}$$

is defined only for nonnegative integers x. It is defined only on discrete points and is not continuous. Hence its derivative does not make sense.