Practice Quiz 1

- This quiz ends at 9 P.M. It contains 10 problems. You have 120 minutes to earn 100 points.
- This quiz is closed book. You may use one $8\frac{1}{2}'' \times 11''$ handwritten crib sheet.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Do not put part of the answer to one problem on the back of the sheet for another problem.
- Do not spend too much time on any problem. Read them all through first and attack them in the order that allows you to make the most progress.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.

Problem	Points	Grade
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total	100	

Name: ______________________________

Problem 0-1. [10 points] Show that any dag (directed acyclic graph) has a vertex with out-degree 0.

Problem 0-2. [10 points] Prove that the sum of the first n squares is $\Theta(n^3)$.

Problem 0-3. [10 points] Let R_1 and R_2 be relations on $A \times B$. Prove that if R_1 and R_2 are transitive, then $R_1 \bigcap R_2$ is transitive.

Problem 0-4. [10 points] Let T be a binary search tree whose keys are distinct, let x be a leaf node, and let y be its parent. Show that $key[y]$ is either the smallest key in T larger than $key[x]$ or the largest key in T smaller than $key[x]$.

Problem 0-5. [10 points] Use Stirling's formula $n! \sim \sqrt{2\pi n}\,(n/e)^n$ to show that

$$\Pi_{i=1}^{n}(2i-1) \sim \sqrt{2}(2n/e)^n.$$

Problem 0-6. [10 points] Prove that every free tree with more than one node has at least two leaves, i.e. nodes of degree 1 (*Hint:* Handshaking Lemma).

Problem 0-7. [10 points] Prove that any graph with $2n$ nodes and no 3-cycles has at most n^2 edges (*Hint:* in inductive step, remove from the graph on $2n+2$ vertices, two adjacent vertices so at most $2n+1$ edges are removed).

Problem 0-8. [10 points]

For each function $f(n)$ along the left side of the table below and each function $g(n)$ across the top, write O, Ω, or Θ in the appropriate space, depending on whether $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, or $f(n) = \Theta(g(n))$. If there is more than one relation between $f(n)$ and $g(n)$, write only the stongest one. *Wrong answers will be penalized, so do not guess unless you are reasonably sure.* The first line is a demo solution.

	$\lg n$	n	n^2
$2^{\lg \lg n}$	Θ	O	O
$4^{\lg n}$			
$\lg(n!)$			
$n / \lg n$			
$n^2 + n + \lg n$			

Problem 0-9. [10 points] Use the substitution method (guess and verify) to determine an asymptotic upper (big O) bound on the recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1, \\ T(n-1) + \lg n & \text{if } n > 1. \end{cases}$$

Problem 0-10. [10 points] Use the method of characteristic equations to solve the linear recurrence

$$f(n) = f(n-1) + f(n-2) + 1 \ ,$$

where $f(1) = 1$ and $f(2) = 2$.