Quiz 2

Problem 1. (10 points) Evaluate the following sum:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^{-(i+j)}$$

Solution. We use the formula for the sum of a geometric series:

$$\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} 2^{-(i+j)} &= \sum_{i=0}^{\infty} 2^{-i} \cdot \left(\sum_{j=0}^{\infty} 2^{-j}\right) \\
&= \sum_{i=0}^{\infty} 2^{-i} \cdot 2 \\
&= 2 \cdot \sum_{i=0}^{\infty} 2^{-i} \\
&= 2 \cdot 2 \\
&= 4
\end{aligned}$$

Problem 2. (10 points) Circle every symbol on the left that could correctly appear in the box to its right. (Thus, for each of the five parts you may circle zero, one, or both of the symbols.)

(**a**) Θ ω $10^n = \boxed{} \left(n^{10}\right)$

(**b**) o O $\dfrac{1}{\log n} = \boxed{} (1)$

(**c**) Ω o $\sqrt{n} = \boxed{} \left((\log n)^5\right)$

(**d**) O Ω $n^2 = \boxed{} \left(3n^2 - 5n + 7\right)$

(**e**) Θ O $n! = \boxed{} (n^n)$

Solution. Underlined symbols should be circled.

$$\begin{array}{cc} \Theta & \underline{\omega} \\ \underline{o} & \underline{O} \\ \underline{\Omega} & o \\ \underline{O} & \underline{\Omega} \\ \Theta & \underline{O} \end{array}$$

Problem 3. (10 points) Caesar conquers an x-square-mile region as follows.

- If $0 < x \leq 1$, then he conquers the whole region in 1 day.
- If $x > 1$, then he spends $\log x$ days partitioning the region into three nonoverlapping parts with sizes $x/6$, $2x/6$, and $3x/6$. Then he conquers each part recursively, one at a time.

Let $C(x)$ be the number of days Caesar needs to conquer an x-square-mile region. Express $C(x)$ with a recurrence equation and suitable boundary condition. Note that x is not necessarily an integer. *Do not solve the recurrence.*

Solution.

$$C(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ \log x + C\left(\frac{1}{6}x\right) + C\left(\frac{2}{6}x\right) + C\left(\frac{3}{6}x\right) & \text{if } x > 1 \end{cases}$$

Problem 4. (15 points) Find a closed-form expression for $T(n)$, which is defined by the following recurrence:

$$\begin{aligned} T(0) &= 0 \\ T(1) &= 1 \\ T(n) &= 5T(n-1) - 6T(n-2) + 6 \qquad \text{for all } n \geq 2 \end{aligned}$$

Solution. The characteristic equation is $x^2 - 5x + 6 = 0$, which has roots $x = 2$ and $x = 3$. Thus, the homogenous solution is:

$$T(n) = A \cdot 2^n + B \cdot 3^n$$

For a particular solution, let's first guess $T(n) = c$:

$$\begin{aligned} c &= 5c - 6c + 6 \\ \Rightarrow c &= 3 \end{aligned}$$

Our guess was correct; $T(n) = 3$ is a particular solution. Adding this to the homogenous solution gives the general solution:

$$T(n) = A \cdot 2^n + B \cdot 3^n + 3$$

Substituting $n = 0$ and $n = 1$ gives:

$$\begin{aligned} 0 &= A + B + 3 \\ 1 &= 2A + 3B + 3 \end{aligned}$$

Solving this system gives $A = -7$ and $B = 4$. Therefore:

$$T(n) = -7 \cdot 2^n + 4 \cdot 3^n + 3$$

Problem 5. (40 points) Solve the three counting problems below. The answers alone are sufficient, but we can only award partial credit if you show your work. You do not need to simplify your answers; you may leave factorials, binomial coefficients, and arithmetic expressions unevaluated.

(a) (15 points) How many seven-card poker hands contain three pairs and no three-of-a-kind or four-of-a-kind?

Solution. There is a bijection with sequences specifying:

- The values of the pairs, which can be chosen in $\binom{13}{3}$ ways.
- The suits of the lowest-value pair, which can be chosen in $\binom{4}{2}$ ways.
- The suits of the middle-value pair, which can be chosen in $\binom{4}{2}$ ways.
- The suits of the highest-value pair, which can be chosen in $\binom{4}{2}$ ways.
- The value of the remaining card, which can be chosen in 10 ways.
- The suit of the remaining card, which can be chosen in 4 ways.

Thus, the number of seven-card poker hands containing three pairs and no three or four-of-a-kind is:

$$\binom{13}{3} \cdot \binom{4}{2}^3 \cdot 10 \cdot 4$$

By the Generalized Product Rule.

(b) (10 points) A shelf holds 17 books in a row. How many ways are there to choose 4 books so that there are at least two books between every pair of chosen books?

Solution. There exists a bijection from 11-bit sequences with exactly four 1's to valid book selections. Map each zero to a non-chosen book, each of the first three 1's to a chosen book followed by two non-chosen books, and the last 1 to a chosen book. For example, the 11-bit sequence:

$$(0, 0, 1, 0, 1, 1, 0, 0, 0, 0, 1)$$

maps to the book selection:

$$\underbrace{n}_{0}\ \underbrace{n}_{0}\ \underbrace{c\,n\,n}_{1}\ \underbrace{n}_{0}\ \underbrace{c\,n\,n}_{1}\ \underbrace{c\,n\,n}_{1}\ \underbrace{n}_{0}\ \underbrace{n}_{0}\ \underbrace{n}_{0}\ \underbrace{n}_{0}\ \underbrace{c}_{1}$$

where c denotes a chosen book and n denotes a non-chosen book. Therefore, the number of valid book selections is $\binom{11}{4}$.

(c) (15 points) The working days in the next year can be numbered 1, 2, 3, ..., 300. I'd like to avoid as many as possible.

- On even-numbered days, I'll say I'm sick.

- On days that are a multiple of 3, I'll say I was stuck in traffic.
- On days that are a multiple of 5, I'll refuse to come out from under the blankets.

In total, how many work days will I *avoid* in the coming year?

Solution. Let D_2 be the set of even-numbered days, D_3 be the days that are a multiple of 3, and D_5 be days that are a multiple of 5. The set of days I can avoid is $D_2 \cup D_3 \cup D_5$. By the Inclusion-Exclusion Rule, the size of this set is:

$$\begin{aligned}
|D_2 \cup D_3 \cup D_5| &= |D_2| + |D_3| + |D_5| \\
&\quad - |D_2 \cap D_3| - |D_2 \cap D_5| - |D_3 \cap D_5| \\
&\quad + |D_2 \cap D_3 \cap D_5| \\
&= \frac{300}{2} + \frac{300}{3} + \frac{300}{5} - \frac{300}{2 \cdot 3} - \frac{300}{2 \cdot 5} - \frac{300}{3 \cdot 5} + \frac{300}{2 \cdot 3 \cdot 5} \\
&= 220
\end{aligned}$$

Problem 6. (15 points) Miss McGillicuddy never goes outside without a collection of pets. In particular:

- She brings 3, 4, or 5 dogs.
- She brings a positive number of songbirds, which always come in pairs.
- She may or may not bring her alligator, Freddy.

Let T_n denote the number of different collections of n pets that can accompany her. For example, $T_6 = 2$ since there are 2 possible collections of 6 pets:

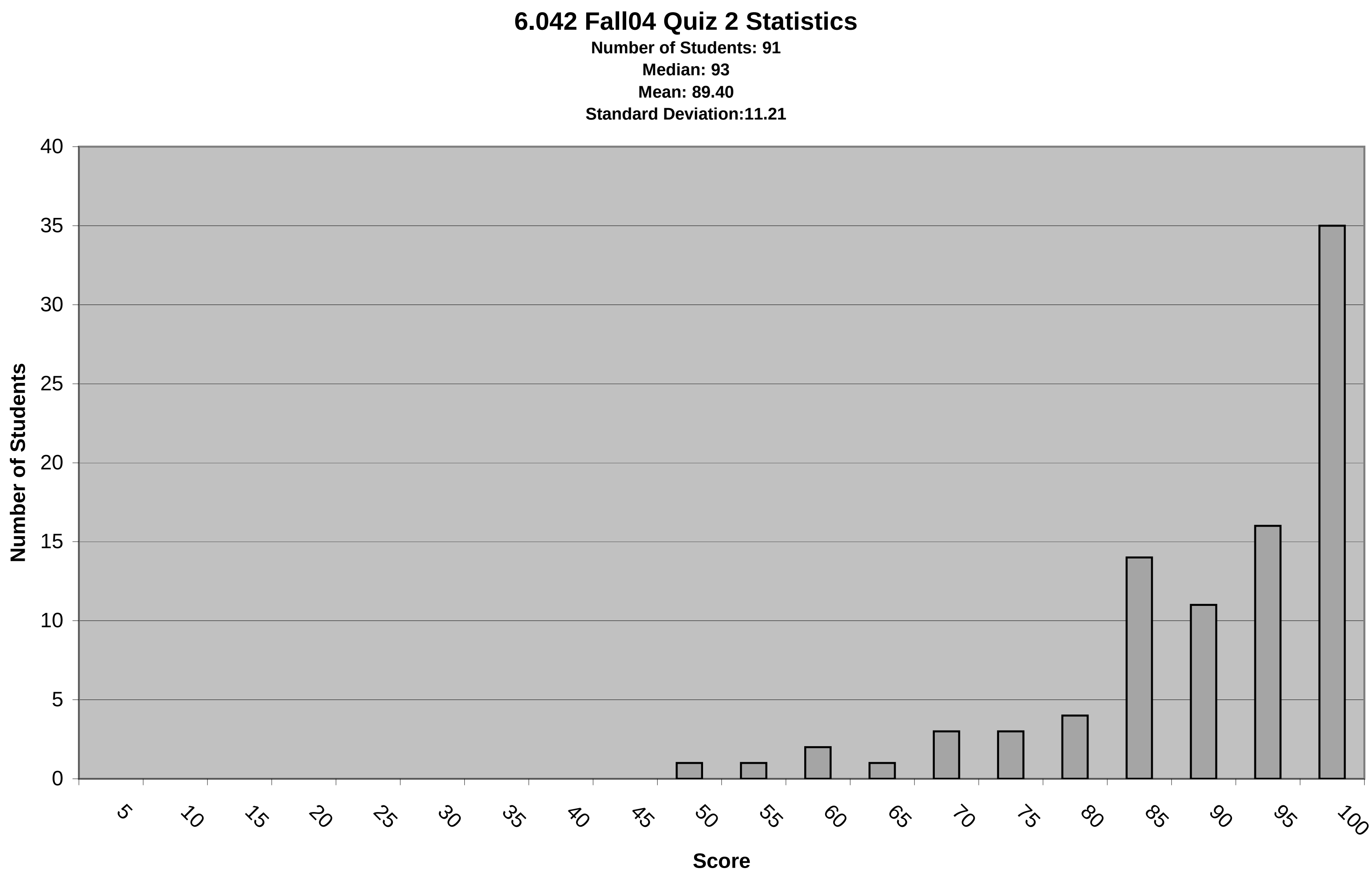
- 3 dogs, 2 songbirds, 1 alligator
- 4 dogs, 2 songbirds, 0 alligators

Give a closed-form generating function for the sequence $\langle T_0, T_1, T_2, T_3, \ldots \rangle$. The answer alone is sufficient, but we can only award partial credit if you show your work.

Solution.

$$\begin{aligned}
T(x) &= \underbrace{(x^3 + x^4 + x^5)}_{\text{collections of dogs}} \cdot \underbrace{(x^2 + x^4 + x^6 + x^8 + \ldots)}_{\text{collections of songbirds}} \cdot \underbrace{(1 + x)}_{\text{collections of gators}} \\
&= (x^3 + x^4 + x^5) \cdot \frac{x^2}{1 - x^2} \cdot (1 + x) \\
&= \frac{x^5 + x^6 + x^7}{1 - x}
\end{aligned}$$

The second equation follows from the formula for the sum of a geometric series. The last step is a simplification, which is not required for full credit.


6.042 Fall04 Quiz 2 Statistics
Number of Students: 91
Median: 93
Mean: 89.40
Standard Deviation:11.21
Number of Students
0
5
10
15
20
25
30
35
40
5
10
15
20
25
30
35
40
45
50
55
60
65
70
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90
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100
Score