Conflict Final

Your name:____

Circle the name of your Tutorial Instructor:

Ching Edmond Karen Kilian Mana Min

- This exam is **closed book**, but you may use two sheets of paper with notes on both sides.
- There are fifteen problems totaling 200 points. Total time is 170 minutes.
- Put your name on the top of **every** page – *these pages may be separated for grading*.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results presented in class or in the assigned reading.
- **Be neat and write legibly**. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	20		
2	10		
3	15		
4	10		
5	10		
6	10		
7	15		
8	10		
9	15		
10	15		
11	15		
12	10		
13	20		
14	15		
15	10		
Total	200		

Problem 1 (20 points). The following short-answer questions are independent and can be answered in any order.

(a) (5 points) Why is the following propositional formula valid?

$$[x \wedge (\neg((\neg x) \longrightarrow y)) \wedge w] \longrightarrow [(u \longrightarrow v) \vee \neg(w \wedge z)]$$

(Hint: Argue by cases according to whether x is true or false.)

(b) (5 points) Does 2^{10} have a multiplicative inverse modulo 3^{11}? Why or why not?

(c) (5 points) Define the *difference* of an ordered pair of integers (x, y) to be $x - y$. Pick any set of 100 integers in the range 1 to 1000. Then there are sure to be at least five different ordered pairs of numbers from the set and all five pairs will have the same ***nonzero*** difference. Briefly explain why.

(d) (5 points) Let C_n be the number of heads obtained in n independent flips of a biased coin that comes up heads with probability $1/3$. Describe the limiting shape of the pdf of C_n/n as n goes to infinity.

Problem 2 (10 points). Sauron wants to conquer Middle Earth. This project involves n tasks. Some of these tasks must be completed before others are begun. For example, the task of *locating* the One Ring must precede the task of *seizing* the ring. Each task can be completed by a horrible creature called a *Ringwraith* in exactly one week. A Ringwraith can complete multiple tasks, but can only work on one at a time.

(a) (2 points) Sauron would like to model this scheduling problem as a directed acyclic graph. What should vertices represent? When should there be a directed edge between two vertices?

(b) (3 points) Sauron is trying to describe various features of his scheduling problem using standard terminology. Next to each feature below, write the number of the corresponding term.

Standard Terminology

1. Transitive closure
2. Topological sort
3. Chain
4. Antichain
5. Size of the largest antichain
6. Size of the smallest antichain
7. Length of the longest chain
8. Length of the shortest chain

1. A set of tasks that can be worked on simultaneously. ______
2. A possible order in which all the tasks could be completed, if only one Ringwraith were available. ______
3. The minimum number of weeks required to complete all tasks, if an unlimited number of Ringwraiths were available. ______

(c) (2 points) Sauron discovers that conquering Middle Earth would take t weeks, if he had an unlimited supply of Ringwraiths, due to the dependencies between tasks.

If Sauron is lucky, he will be able to get by with a smallish crew of Ringwraiths and still be able to conquer the world in t weeks. Write a simple formula involving n and t for the smallest number of Ringwraiths that could possibly be able to complete all n tasks in t weeks.

(d) (3 points) On the other hand, if Sauron is unlucky, he may need a large crew of Ringwraiths in order to conquer Middle Earth in t weeks. Write a simple formula involving n and t for the largest number of Ringwraiths that Suaron would need in order to be sure of completing all n tasks in t weeks – no matter how unlucky he was.

Problem 3 (15 points). The Massachusetts Turnpike Authority has a new policy for traffic control on the Zakim bridge. The bridge will be one way; cars will pay tolls both on entering and exiting the bridge, but the tolls will be different. In particular, a car will pay \$5 to enter onto the bridge and will pay \$3 to exit. To be sure that there are never too many cars on the bridge, the Authority will let a car onto the bridge only if *the difference between the amount of money currently at the entry toll booth minus the amount at the exit toll booth is strictly less than \$2500*.

The Authority has asked their engineering consultants to verify that this policy will keep the number of cars from exceeding 500. The consultants have decided to model this scenario with a state machine whose states are triples of natural numbers, (A, B, C), where

- A is an amount of money at the entry booth,
- B is an amount of money at the exit booth, and
- C is a number of cars on the bridge.

Any state with $C > 500$ is called a *collapsed* state, which the Authority dearly hopes to avoid. There will be no transition out of a collapsed state.

Since the toll booth collectors may need to start off with some amount of money in order to make change, and there may also be some number of "official" cars already on the bridge when it is opened to the public, the consultants must be ready to analyze the system started at *any* state, not necessarily with a start state $(0, 0, 0)$. (However, you should assume that even official cars pay tolls when the system is running.)

(a) (3 points) Give a mathematical model of the Authority's system for letting cars on and off the bridge by specifying a transition relation between states of the form (A, B, C) above.

(b) (5 points) Characterize each of the following derived variables

B	WI
$A + B$	SI
$5C - A$	
$3A - 5B$	
$B + 3C$	
$15C - 3A + 5B$	
$A - 4C - B$	

as one of the following

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

by entering the corresponding letter-code next to the variable.

(c) (7 points) Define a simple Invariant, P, of the transition system which is satisfied by the state $(0, 0, 0)$ and is not satisfied by any collapsed state. That is, P is a predicate on states that is preserved by transitions from *any* state to another. In addition, $P(0, 0, 0)$ is true, and $P(A, B, C)$ is not true whenever $C > 500$.

No proof is required.

Problem 4 (10 points). A couple wants to borrow d dollars so that they can purchase a new condo. At the end of each year for the next n years, one of them will drop by the bank to make a payment on the loan. The banker will immediately invest this returned money at an r% annual rate of return. The banker wants to wind up with at least as much money as a result of the couple's payments as he would if he had invested the whole d dollars at r% for n years starting today.

Write a closed form formula in d, n and r for the least annual payment the banker might ask from the couple.

Problem 5 (10 points). Here are the solutions to the next 10 problem parts, in no particular order.

$$n^m \qquad m^n \qquad \frac{n!}{(n-m)!} \qquad \binom{n-1+m}{m} \qquad \binom{n-1+m}{n} \qquad 2^{mn}$$

(a) (1 point) How many ways are there to put m indistinguishable balls into n distinguishable urns? ________

(b) (1 point) How many ways are there to put m distinguishable balls into n distinguishable urns? ________

(c) (1 point) How many solutions over the natural numbers are there to the equation $x_1 + x_2 + \ldots + x_n = m$? ________

(d) (1 point) How many $\sqrt{n} \times \sqrt{n}$ matrices are there with entries drawn from $\{1, 2, \ldots, m\}$? ________

(e) (1 point) How many different subsets of the set $A \times B$ are there, if $|A| = m$ and $|B| = n$? ________

(f) (1 point) How many functions are there from set A to set B, if $|A| = n$ and $|B| = m$? ________

(g) (1 point) How many m-letter words can be formed from an n-letter alphabet, if no letter is used more than once? ________

(h) (1 point) How many m-letter words can be formed from an n-letter alphabet, if letters can be reused? ________

(i) (1 point) How many relations are there from set A to set B, where $|A| = m$ and $|B| = n$? ________

(j) (1 point) How many injections are there from set A to set B, where $|A| = m$ and $|B| = n$? ________

Problem 6 (10 points). We consider a hypothetical Institute with only two departments: EECS and Math, and consider the experiment where we pick a random applicant for admission to one of these departments. Define the following events:

$$\begin{aligned}
A &::= \text{applicant is accepted,} \\
F &::= \text{applicant is female,} \\
M &::= \text{applicant is male,} \\
MATH &::= \text{applicant applied to Math,} \\
EECS &::= \text{applicant applied to EECS.}
\end{aligned}$$

Assume that all applicants are either male or female, and that no applicant applied to both departments.

(a) (5 points) The Institute is sued for gender discrimination, with the plaintiff observing that in both EECS and Math, the probability that a woman is accepted is less than the probability that a man is accepted. This observation can be precisely expressed with two inequalities between conditional probabilities involving the events above. State these inequalities.

(b) (5 points) The Institute's defense attorneys retort that overall among all applicants, a woman is *more* likely to be accepted than a man. Express this observation as an inequality between conditional probabilities involving the events above.

Problem 7 (15 points). Let $\mathcal{S}$ be the set $\{a, b, c\}$. Suppose that we construct a binary relation R on $\mathcal{S}$ with a randomized procedure. In particular, for each $x, y \in \mathcal{S}$, the relation xRy holds with probability p and fails to hold with probability $q ::= 1 - p$ mutually independently. Write formulas in terms of p and q for the probability of each of the following events.

A correct answer gets full credit, but you can get partial credit only if you include an explanation with your answer.

(a) **(2 points)** R is reflexive.

(b) **(5 points)** R is symmetric.

(c) **(3 points)** R is an equivalence relation with the equivalence classes $\{b\}$ and $\{a, c\}$.

(d) **(5 points)** R is an equivalence relation.

Problem 8 (10 points). You knock on the door of a home with two children. Each child is a boy or girl with equal probability, and each child is equally likely to open the door.

A sample space for this experiment has outcomes that are triples whose first element is either B or G for the sex of the elder child, likewise for the second element and the sex of the younger child, and whose third coordinate is E or Y indicating whether the *e*lder child or *y*ounger child opened the door. For example, $(\mathtt{B}, \mathtt{G}, \mathtt{Y})$ is the outcome that the elder child is a boy, the younger child is a girl, and the girl opened the door.

(a) (2 points) Let T be the event that the household has two girls, and O be the event that a girl opened the door. List the outcomes in

- T: ____

- O: ____

(b) (2 points) What is the probability $\Pr\{T \mid O\}$, that both children are girls, given that a girl opened the door?

(c) (6 points) Identify exactly where the error is in the following argument.

If a girl opens the door, then we know that there is at least one girl in the household. The probability that there is at least one girl is

$$1 - \Pr\{\textit{both children are boys}\} = 1 - (1/2 \times 1/2) = 3/4. \tag{1}$$

So,

$$\begin{aligned}
&\Pr\{T \mid \textit{there is at least one girl in the household}\} && (2)\\
&= \frac{\Pr\{T \cap \textit{there is at least one girl in the household}\}}{\Pr\{\textit{there is at least one girl in the household}\}} && (3)\\
&= \frac{\Pr\{T\}}{\Pr\{\textit{there is at least one girl in the household}\}} && (4)\\
&= (1/4)/(3/4) = 1/3. && (5)
\end{aligned}$$

Therefore, given that a girl opened the door, the probability that there are two girls in the household is $1/3$.

Problem 9 (15 points). Here are seven propositions:

$$
\begin{array}{rcrcr}
x_1 & \vee & x_3 & \vee & \neg x_7 \\
\neg x_5 & \vee & x_6 & \vee & x_7 \\
x_2 & \vee & \neg x_4 & \vee & x_6 \\
\neg x_4 & \vee & x_5 & \vee & \neg x_7 \\
x_3 & \vee & \neg x_5 & \vee & \neg x_8 \\
x_9 & \vee & \neg x_8 & \vee & x_2 \\
\neg x_3 & \vee & x_9 & \vee & x_4
\end{array}
$$

Note that each proposition is the OR of three expressions of the form x_i or $\neg x_i$. Moreover, the variables in the three expressions are different. Suppose that we assign true/false values to the variables $x_1, \ldots, x_9$ independently and with equal probability. Justify your answers to the following questions.

(a) (3 points) What is the probability that the first proposition is false?

(b) (4 points) What is the expected number of false propositions?

(c) (6 points) Use Markov's Inequality to derive a number $p > 0$ such that all the propositions are true with probability at least p.

(d) (2 points) Suppose that we generate random assignments until we find one that makes all the propositions true. Give an upper bound on the expected number of assignments that we generate. You may express your answer in terms of the probability, p, from the previous part.

Problem 10 (15 points). There are 300 different episodes of the television program *Law and Order*, of which 50 are *really great* and 250 are *pretty good*. One episode is rerun each night. Assume that this episode is selected uniformly at random, independent of all preceding episodes.

(a) (3 points) What is the expected number of episodes you will watch until you see one really-great episode?

(b) (6 points) Suppose that I've seen k of the really-great episodes where $0 \leq k < 50$. What is the expected number of additional episodes I must watch in order to see a really-great episode I have not seen yet?

(c) (6 points) What is the expected number of episodes that I must watch in order to see *all* of the really-great episodes? Your answer should be a closed-form expression involving the nth Harmonic number,

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{n}.$$

Problem 11 (15 points). Suppose successive digits from zero to nine are generated independently until the four digit sequence **9999** appears. What is the expected number of digits generated? Briefly explain your answer.

Hint: Parse the sequence into consecutive "tries to get 9999", where a "try" is either a single digit other than 9, or a 9 followed by a single digit other than 9, etc. For example, the sequence 92349982999769999 parses into eight tries:

$$/92/3/4/998/2/9997/6/9999/.$$

Problem 12 (10 points). Let $G_1, G_2, G_3, \ldots,$ be an infinite sequence of independent random variables with the same distribution and the same expectation, μ, and let

$$f(n, \epsilon) ::= \Pr\left\{\left|\frac{\sum_{i=1}^{n} G_i}{n} - \mu\right| \leq \epsilon\right\}.$$

The Weak Law of Large Numbers can be expressed as a logical formula of the form:

$$\forall \epsilon > 0\, Q_1\, Q_2 \ldots\ [f(n, \epsilon) \geq 1 - \delta]$$

where $Q_1 Q_2 \ldots$ is a sequence of quantifiers from among:

- $\forall n_0$
- $\forall n \geq n_0$
- $\forall \delta > 0$
- $\forall \delta \geq 0$
- $\exists n_0$
- $\exists n \geq n_0$
- $\exists \delta \geq 0$
- $\exists \delta > 0$

Here the n and n_0 range over natural numbers, and δ and ϵ range over real numbers.

Write out the proper sequence $Q_1 Q_2 \ldots$ ______________________________________

Problem 13 (20 points). Let T_n be the graph consisting of n consecutive triangles arranged as follows:

```
      ________________________________ ..._ __________
     /\        /\        /\        /          \        /
    /  \      /  \      /  \      /    ...     \      /
   /    \    /    \    /    \    /              \    /
  /______\/______\/______\/___  ...  ______\/
```

(a) (4 points) Each edge in T_n is colored red with probability r and blue with probability $b ::= 1 - r$ mutually independently. A triangle is *monochromatic* if its edges are all blue or all red. What is the probability, m, that a particular triangle is monochromatic?

(b) (3 points) Let I_T be an indicator random variable for the event that a given triangle, T, is monochromatic.What is $\mathrm{E}[I_T]$? ______ What is $\mathrm{Var}[I_T]$? ______

You may state your answer in terms of the probability, m, from the previous problem part.

(c) (4 points) Let T and T' be any two different triangles, and suppose that $r = 1/2$. Show that the random variables I_T and $I_{T'}$ are independent.

(d) (4 points) Let M be the random variable equal to the total number of monochromatic triangles in the graph. If $r = 1/2$, what is $\text{Var}\,[M]$?

(e) (5 points) Prove that

$$\lim_{n\to\infty} \Pr\left\{|M - \mathrm{E}[M]| \geq n/1000\right\} = 0.$$

Problem 14 (15 points). A herd of cows is stricken by an outbreak of *cold cow disease*. The disease lowers the normal body temperature of a cow, and a cow will die if its temperature goes below 90 degrees F. The disease epidemic is so intense that it lowered the average temperature of the herd to 85 degrees. Body temperatures as low as 70 degrees, **but no lower**, were actually found in the herd.

(a) (10 points) Based solely on the information above, state a **largest possible** lower bound on the probability that a randomly chosen cow from herd will have a temperature too low to survive.

(b) (5 points) Suppose there are 400 cows in the herd. Give an example set of temperatures for the cows so that the probability that a randomly chosen cow will have a fatal temperature is as small as possible.

Problem 15 (10 points). In the class Notes, we analyzed the Gambler's Ruin game where the Gambler's initial stake was $\$n$, and he made independent bets with probability p of winning \$1 and probability $q ::= 1 - p$ of losing \$1, until his stake was reduced to zero or he reached his goal of \$T. We let w_n be the probability that the Gambler *wins* by reaching his goal before running out of money. The problem of finding a formula for w_n was reduced to solving the following quadratic equation in c:

$$c^2 - \frac{c}{p} + \frac{q}{p} = 0.$$

Now we consider a slight alteration of the game: the probability of winning each bet is still p, but instead of winning \$1 on a winning bet, the Gambler wins \$2. He still loses only \$1 on a losing bet. Finding a formula for w_n in this case reduces to solving a cubic equation in c:

$$a_3c^3 + a_2c^2 + a_1c + a_0 = 0.$$

Fill in values for the coefficients of this cubic formula:

a_3______,

a_2______,

a_1______,

a_0______.

A correct answer gets full credit, but you can get partial credit only if you include an explanation with your answer.