Quiz 1

Your name:____

Circle the name of your Tutorial Instructor:

Adrian Georgi Josh Karen Lee Min Nikos Tina

- This quiz is **closed book**. There is an Appendix with standard definitions.
- There are five (5) problems totaling 100 points. Total time is 110 minutes.
- Put your name on the top of **every** page – *these pages may be separated for grading*.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- You may assume any of the results presented in class or in the lecture notes.
- **Be neat and write legibly**. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

This is where the grades table is supposed to be printed. Compile the paper once more to get it right. See `course.sty` for details.

Problem 1 (20 points). Suppose $S(n)$ is a predicate on natural numbers, n, and suppose

$$\forall k \in \mathbb{N}\; S(k) \longrightarrow S(k+2). \tag{1}$$

If (**??**) holds, some of the assertions below must *always* (**A**) hold, some *can* (**C**) hold but not always, and some can *never* (**N**) hold. Indicate which case applies for each of the assertions by **circling** the correct letter.

(a) (2 points) **A C N** $\forall n \geq 0\; S(n)$

(b) (2 points) **A C N** $\neg S(0) \wedge \forall n \geq 1\; S(n)$

(c) (2 points) **A C N** $\forall n \geq 0\; \neg S(n)$

(d) (2 points) **A C N** $(\forall n \leq 100\; S(n)) \wedge (\forall n > 100\; \neg S(n))$

(e) (2 points) **A C N** $(\forall n \leq 100\; \neg S(n)) \wedge (\forall n > 100\; S(n))$

(f) (2 points) **A C N** $S(0) \longrightarrow \forall n\; S(n+2)$

(g) (2 points) **A C N** $S(1) \longrightarrow \forall n\; S(2n+1)$

(h) (2 points) **A C N** $[\exists n\, S(2n)] \longrightarrow \forall n\; S(2n+2)$

(i) (2 points) **A C N** $\exists n\, \exists m > n\, [S(2n) \wedge \neg S(2m)]$

(j) (2 points) **A C N** $[\exists n\, S(n)] \longrightarrow \forall n\, \exists m > n\, S(m)$

Problem 2 (15 points). Attila the Hun is planning another excursion into a Roman village. This requires a number of tasks, each of which takes him one minute to complete. The prerequisites associated with these tasks are listed below.

ABBRV.	TASK	PREREQUISITES
C	Assemble the barbarians	
N	Plunder the village	B
D	Get shots for his own cat, also named Emilios	B
B	Blow the trumpet	
T	Sell T-shirts: "I got A LOT more than just this lousy t-shirt."	N
Q	Grade the 6.042 quiz	S
G	Cook a feast	D,N
S	Burn the village	C,N

(a) (4 points) Draw the Hasse diagram for the tasks and their prerequisites.

(b) (2 points) Attila has decided that since his barbarians are quite smart and he has so many, he can get as many tasks done at a time as he wishes. What is the minimum amount of time required for him to finish the excursion? ________

It turns out that Attila's graph is actually far more complicated than the one above. In fact, he doesn't know what the actual diagram because his Scribe forgot to tell him. All Attila has been told is that he must complete n tasks, each of which takes 1 minute, and that the minimum amount of time required to finish is t minutes. Without knowing anything more about the actual graph, Attila is trying to figure out how many barbarians to recruit. A barbarian can only complete one task in 1 minute, but has the stamina to work for days on end. Let n and t be fixed, and $n > t > 1$.

(c) (4 points) Write a simple formula in n and t for the smallest number of barbarians Attila can recruit in order to be *guaranteed* to finish in t minutes. ________

(d) (5 points) Write a simple formula in n and t for the smallest number of barbarians he could recruit and still *possibly* finish the job in t minutes. (That is, if he recruits fewer barbarians, he will never be able to finish in the minimum number, t, of minutes, no matter how "favorable" graph turns out to be.) ________

Problem 3 (15 points). In this problem, let R be a binary relation on a set A, and S be a binary relation from A to a set B. Indicate whether each of the following statements is **True** or **False**. For the false ones, ***describe a counterexample***[1].

(a) (3 points) If $R = R^*$, then R is a transitive relation. ______

(b) (3 points) If $R = R^*$, then R is an equivalence relation. ______

(c) (3 points) If R is an equivalence relation, then $R = R^*$. ______

(d) (3 points) If $A = B$, then $S \circ R = R \circ S$. ______

(e) (3 points) If $R = R^{-1}$ and R is nonempty, then R is reflexive. ______

[1] No explanation required for statements that are True.

Problem 4 (25 points). Recall that a k-coloring of a simple graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color, and no more than k colors are used. In class we proved

Lemma. If a graph has maximum degree k, then it is $k+1$-colorable,

Consider the following variation of this Lemma:

False Claim. *A graph with maximum degree k that also has a vertex of degree less than k, is k-colorable.*

(a) (5 points) Give a counterexample to the False Claim when $k = 2$.

Consider the following proof of the False Claim:

False proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Every graph with n vertices and maximum degree k that also has a vertex of degree less then k, is k-colorable."

Base case: (n=1) The graph has only one vertex of degree zero, so $P(1)$ holds vacuously.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with maximum degree k that has a vertex, v, of degree less than k.

Remove the vertex v to produce a graph G_n. Removing v reduces the degree of all vertices adjacent to v by 1. Therefore G_n must contain at least one vertex with degree less than k. Also the maximum degree of G_n is at most k. If the maximum degree of G_n is less than k, then by the Lemma above, G_n is k-colorable. Otherwise G_n has maximum degree k and a vertex of degree less than k, so by our induction hypothesis, G_n is k-colorable. So in any case, G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So there will be a color left over among the k colors that can be assigned to v. Hence G_{n+1} is k-colorable. □

(b) (5 points) Identify the exact sentence where the proof goes wrong (underlining or circling the sentence is sufficient).

(c) (5 points) Briefly explain why the sentence is incorrect.

(d) (5 points) The False Claim can be made true by adding an additional assumption about the graph. Which of the following is the most general asumption that will make the False Claim true?[2]

1. The graph is a line graph.
2. The graph has only even length cycles.
3. The graph is connected.
4. The graph does not contain a complete graph on k vertices.
5. The graph has no node of degree zero.
6. The graph has a Hamiltonian cycle.
7. $k < 2$.

[2] By "most general," we mean it is implied by all the other assumptions that verify the False Claim. For example, the assumption that G is a line graph is more general than the assumption that G is a line graph with 3 vertices.

(e) (5 points) Assuming that G_{n+1} satisifes the additional assumption from part (d), both the False Claim and the sentence that was incorrect from part (b) become correct. But now another sentence in the proof becomes incorrect and requires fixing. Indicate the new incorrect sentence and briefly explain what's wrong. (You are *not* expected to suggest a fix.)

Problem 5 (25 points). One of the three monks working on the famed Towers of Hanoi project recently rubbed his pained back and burst out, "Yo! What are we doing? This is for chumps! Let's punt!" But before wandering off to start up fast food joints, they must evenly divide the monastery's collection of prayer beads.

Initially, monk A has 5 beads, monk B has 3 beads, and monk C has 4 beads. The monastic order has strict rules regarding the exchange of prayer beads. Only the following transactions are allowed.

1. Monk B may give a bead to monk A at any time.
2. If C has an odd number of beads, then monk A may give a bead to monk B.
3. If C has an even number of beads, then monk C may give or take a bead from either monk A or monk B.
4. If monk A has at least two more beads than monk B, then monk C may give or take a bead from either monk A or monk B.

(a) (10 points) Model the situation with a state machine. Define the set of states, the set of start states, and the set of transitions.

(b) (5 points) Describe a sequence of steps (transitions) leading to a state where monk A has 0 beads, monk B has 9 beads, and monk C has 3 beads.

(c) (4 points) A clever TA tells you that the following predicate is an invariant.

$$(C \text{ has an odd number of beads}) \vee (A \text{ has more beads than } B)$$

Assuming she is correct, prove that the monks can not reach the state where every monk has 4 beads.

(d) (6 points) Now prove that the TA was correct in her assumption that the predicate is an invariant.