Problem Set 5

Due: Tuesday, March 9.

Problem 1 (16 points) Find closed-form expressions for the following sums.

(a)

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} 2^{-(i+2j+3k)}$$

(b)

$$\sum_{j=1}^{n}\sum_{i=0}^{\infty} j^{2/3}\cdot\left(1-\frac{1}{2j^{1/3}}\right)^{i}$$

Problem 2 (10 points) The expression $\binom{n}{k}$ is read "n choose k" and represents the quantity

$$\frac{n!}{k!\,(n-k)!}.$$

This expression comes up frequently in probability and combinatorics and will be used extensively later in this course. Suppose that n is even. Prove that

$$\binom{n}{n/2} \quad = \quad \Theta(2^n/\sqrt{n}).$$

Use the following form of Stirling's formula:

$$n! \quad \sim \quad \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

Problem 3 (10 points) Let

$$S_n ::= \sum_{k=1}^{n} 1/(2k-1).$$

(a) Find a closed form expression for S_n in terms of the harmonic numbers H_n and H_{2n}.

(b) Use an approximation for the harmonic numbers to show that

$$S_n = \ln(2\sqrt{n}) + O(1).$$

Problem 4 **(10 points)** Use the integral method to find a Θ bound on the following sum.

$$\sum_{i=1}^{n} i \ln i$$

Problem 5 **(20 points)** In analyzing algorithms, one often must determine the asymptotic behavior of nasty functions like those shown below.

$$\begin{array}{ccc}
n! & (\ln n)^{300} & 2n^{\ln \ln n} \\
n^n & n(\ln n)^4 & n^{1.01}/2 \\
7n^2+3 & \sqrt{n}^{\sqrt{n}} & 4^n \\
(n/e)^n & (7n^3-17n^2)/(n+128) & (\ln n)^{\ln n}
\end{array}$$

Arrange these functions in a list so that each is Θ or O of the next function in the list. For each pair of consecutive functions in your list, indicate whether the stonger Θ relationship holds or only the weaker O relationship. No proofs are required.

Problem 6 **(10 points)**

Recently, David Bowie, a British music star, sold a portion of the future profits from his music for \$55 million dollars to Prudential Insurance. This deal spawned others.

(a) Billy Joel wants to sell his future profits for \$200 million today. Suppose that he can generate a fixed profit at the end of every year for the next 20 years. If the inflation rate is $p = 0.03$, how large must Billy Joel's annual profit be for investors to make a 10% return?

(b) The Eagles figure that they can generate at least \$25 million from their music at the end of each year for eternity. What should an investor who wants a 15% return pay today for the Eagles contract?

Problem 7 **(24 points)** Shady finance people often assert that a loan can be paid off more quickly by making half-size payments twice as often. The theory is that frequent payments prevent interest from building up. Suppose that I take out a loan for L dollars, and at the end of every month, the bank charges me interest on the loan at rate i.

(a) First, let's consider a "frequent payment" repayment strategy. Suppose that I pay P dollars to the bank every month, right after the bank charges me interest. (So, after 1 month I owe $L(1+i) - P$ dollars, and after 2 months I owe $L(1+i)^2 - P \cdot (1+i) - P$ dollars.) Give a closed-form expression for the number of months until the loan is repaid.

(Assume that interest charges are smaller than my monthly payment; that is, $iL < P$. Do not worry about rounding to an integral number of months.)

(b) Now let's consider an "infrequent payment, late start" repayment strategy. Suppose that I pay $2P$ dollars to the bank every two months, making the first payment two months after I take out the loan. (Thus, after 2 months I owe $L(1+i)^2 - 2P$ dollars, and after 4 months I owe $L(1+i)^4 - 2P \cdot (1+i)^2 - 2P$ dollars.) Give a closed-form expression for the number of months until the loan in repaid.

(c) Finally, let's consider an "infrequent payment, early start" strategy. As before, I pay $2P$ dollars to the bank every two months, but now I make the first payment one month after I take out the loan. (Thus, after 1 month I owe $L(1+i) - 2P$ dollars, and after 3 months I owe $L(1+i)^3 - 2P \cdot (1+i)^2 - 2P$ dollars.) Give a closed-form expression for the number of months until the loan is repaid.

(d) Suppose that I take a loan of $L = \$100,000$. The monthly interest rate is $i = 0.005$. I either pay the bank $P = \$600$ every month or $2P = \$1,200$ every two months, depending on the repayment strategy that I choose. How long will repayment take for each of the three strategies?

(e) In 50 words or fewer, explain the merits of frequent and infrequent payment strategies.