Problem Set 5

Due: 2:35pm in lecture, Thursday, October 12, 2000.

Be sure to indicate your TA's name clearly on the solutions your turn in.

Self-study: Simple algorithms and pseudocode Read: Rosen Section 2.1 Self-study problems (not to be handed in): 1, 3, 5, 7, 9, 11, 19

Reading: Lecture notes; Rosen 2.1, 3.5 (skim)

Policy on collaboration If you collaborated with someone on the problem sets, you should mention their names on top of the problem set. If you did not collaborate with anyone, you should explicitly say so. As a reminder, the write-up of your solutions should be entirely your own.

Problem 1 In ancient times, the downtown section of Verona was laid out in a 1000 by 1000 meter square. Romeo lived at grid point $(0, 0)$ of the square, and Juliet lived at grid point $(0, 1000)$. Of course, Romeo and Juliet wanted to meet, at *exactly* the same point in the square. However, both were restricted by their rather strange families to move according to very particular rules: Romeo could only move 3 units, parallel to either the x axis or y axis or at a 45 degree angle to both axes. Juliet could only move 3 units, and only parallel to either the x axis or y axis.

(a) Model Romeo and Juliet and their possible moves as a (single) state machine. That is, define the states Q, the start state Q_0, and the transitions δ.

(b) Can Romeo and Juliet meet by some sequence of moves? If yes, then give such a sequence of moves. If no, then prove formally (using an invariant) that no sequence of moves can yield a meeting.

Problem 2 In the late 1960s, the military junta that ousted the government of the small republic of Miranda outlawed multiplication and division by any number other than 3. When democracy was reinstated 30 years later, people were interested in rediscovering the lost art of multiplication. A young inventor came up with the following algorithm to multiply any two natural numbers x and y using only multiplication and division by 3:

procedure *multiply*(x, y: natural numbers)
$r := x$;

$s := y$;
$a := 0$;
do until $s = 0$
    if $s \equiv 0 \pmod 3$ **then**
        $r := 3r$;
        $s := s/3$;
    else if $s \equiv 1 \pmod 3$ **then**
        $a := a + r$;
        $r := 3r$;
        $s := (s - 1)/3$;
    else
        $a := a + 2r$;
        $r := 3r$;
        $s := (s - 2)/3$;
return a;

(a) Model the algorithm as a state machine (Q, Q_0, δ).

(b) List the sequence of steps that appears in the execution of the algorithm for inputs $x = 5$ and $y = 10$.

(c) Prove that the algorithm gives the correct answer if it terminates, that is, that $s = 0$ implies $a = xy$. (*Hint*: Use induction on the number of steps in the execution. You will have to strengthen the inductive hypothesis.)

(d) Prove that the algorithm terminates.

Problem 3 Shah Shahryar's wife betrays him with a slave and he vows revenge on all women. Each night, he will marry a new bride and the next morning have her beheaded. One night comes beautiful Sheherezade and presents the Shah with a game:

You have three numbers: 1, 1 and 2. At each time, you are allowed to replace any two numbers x, y with $\frac{x+y}{\sqrt{2}}$, $\frac{x-y}{\sqrt{2}}$, but you must not change your third number. You will win only if, using this rule, you can obtain the numbers a, b and c.

(a) Model Sheherezade's game as a state machine. (*Note*: The representation is a little tricky because the numbers are not ordered.)

For each of the following choices of a, b and c, say whether or not the Shah can win the game. If yes, show a sequence of winning steps. If no, provide a proof using an invariant.

(b) $(a, b, c) = (\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}+1}{2}, \frac{\sqrt{2}-1}{2})$

(c) $(a, b, c) = (\frac{\sqrt{3}-1}{2}, \frac{\sqrt{3}+1}{2}, 7)$

(d) $(a, b, c) = (1, 2, 2)$

Problem 4 Consider the matching problem from lecture.

(a) For a particular value of n, give an example of a stable match between n boys and n girls where no person gets their first choice.

(b) Can a stable match where no person gets their first choice be produced by the matching algorithm described in class? If yes, state an example and show why the example works. If not, prove using an invariant.

Problem 5 Consider an instance of the matching problem with n boys and n girls. Call a person *unlucky* if he or she is matched up with one of his or her $\lfloor \frac{n}{2} \rfloor$ last choices. In this problem, we will prove the following:

Theorem The matching algorithm from class never produces a matching in which every person is unlucky.

Fix an execution of the matching algorithm. Define the variables $B_i, G_i, i \in \{1, 2, \ldots, n\}$ as follows:

$B_i = j$ if the i-th boy is currently courting the j-th girl on his list

G_i is the number of boys that the i-th girl has rejected.

(a) Show that $\sum_{i=1}^{n} B_i - \sum_{i=1}^{n} G_i$ is preserved at each step of the matching algorithm.

(b) Formulate an invariant that you can use to prove the theorem, and show that your invariant is correct.

(c) Use your invariant to prove the theorem.