Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$ Start state is

   0 is a mult

2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$
   1. Fix $n \geq 0$.
   2. Assume $P(n)$: after n steps, both jars have multi
   3. $P(n+1)$: after $n+1$ steps, both jars have multi

      ???
   4. QED Implication,
3. QED Induction

3. $P(n+1)$, that is, after $n+1$ steps, both jars have r
   1. Fix $q_0, q_1, \ldots, q_{n+1}$, an execution with $n+1$ steps
   2. In q_n, both jars have multiples of 3. Inductive hyp
   3. In q_{n+1}, both jars have multiples of 3.
      1. If the last move is to fill the little jar, then ...
      2. If the last move is to fill the big jar, then...
      3. If the last move is to empty the little jar...
      4. If the last move is to empty the big jar...
      5. Pour from little to big...
      6. Pour from big to little...
      7. QED Cases
   4. QED UG

Prove: $\forall n \geq 0(P(n))$

1. (Base) $P(0)$ Start state is $(0, 0)$, $0 + 2 \times 0$ is a mult
2. (Inductive step) $\forall n \geq 0(P(n) \Rightarrow P(n+1))$
    1. Fix $n \geq 0$.
    2. Assume $P(n)$, that is, after n steps, $x + 2y$ is a multiple of 7
    3. $P(n+1)$, that is, after $n+1$ steps, $x + 2y$ is a multiple of 7
        1. Fix $q_0, q_1, \ldots, q_{n+1}$, an execution with $n+1$ steps.
        2. In q_n, $x + 2y$ is a multiple of 7. Inductive hypothesis.
        3. In q_{n+1}, $x + 2y$ is a multiple of 7.
            1. If the last move is $(+2, -1)$ then in q_{n+1}, $x + 2y$ is a m
            2. If the last move is $(-2, +1)$ then in q_{n+1}, $x + 2y$ is a m
            3. If the last move is $(+1, +3)$ then in q_{n+1}, $x + 2y$ is a
            4. If the last move is $(-1, -3)$ then in q_{n+1}, $x + 2y$ is a m
            5. QED Cases
    4. QED Implication, UG
3. QED Induction

1. If last move is $(+2, -1)$ then in q_{n+1}, $x + 2y$ is a mu

    1. Assume the last move is $(+2, -1)$.
    2. Let x and y denote the values in state q_n,
       x' and y' the values in state q_{n+1}.
    3. $x' = x + 2$, $y' = y - 1$. Definition of the m
    4. $x + 2y \cong 0 mod 7$ Inductive hypothes
    5. $x' + 2y' = (x + 2) + 2(y - 1) = x + 2y \cong 0 mod 7$
       Algebra
    6. In q_{n+1}, $x + 2y$ is a multiple of 7.
    7. QED Implication

3. If last move is $(+1, +3)$ then in q_{n+1}, $x + 2y$ is a m
   1. Assume the last move is $(+1, +3)$.
   2. Let x and y denote the values in state q_n,
      x' and y' the values in state q_{n+1}.
   3. $x' = x + 1$, $y' = y + 3$. Definition of the m
   4. $x + 2y \cong 0 mod 7$ Inductive hypothes
   5. $x' + 2y' = (x + 1) + 2(y + 3) = x + 2y + 7 \cong 0 mo$
      Algebra
   6. In q_{n+1}, $x + 2y$ is a multiple of 7.
   7. QED Implication

3. $P(n+1)$, that is, after $n+1$ steps, $GCD(x,y) = GC$
   1. Fix $q_0, q_1, \ldots, q_{n+1}$, an execution with $n+1$ steps
   2. In q_n, $GCD(x,y) = GCD(a,b)$. Inductive hyp
   3. In q_{n+1}, $GCD(x,y) = GCD(a,b)$.
      1. If $x > y$ in q_n then in q_{n+1}, $GCD(x,y) = GCD($
         1. Assume $x > y$ in q_n.
         2. Let x and y denote the values in state q_n, x' and y' the values in state q_{n+1}.
         3. $x' = x - y$, $y' = y$ Definition of
         4. $GCD(x,y) = GCD(a,b)$. Inductive hyp
         5. $GCD(x',y') = GCD(x-y,y)$ Plug in
         6. $GCD(x-y,y) = GCD(x,y)$ Number theo
         7. $GCD(x',y') = GCD(a,b)$ Algebra
         8. In q_{n+1}, $GCD(x,y) = GCD(a,b)$
         9. QED Implication
      2. If $x < y$ in q_n then in q_{n+1}, $GCD(x,y) = GCD($
         Analogous to
      3. QED Cases

Prove: Any live execution of GCD contains some state

1. Assume not, that is, $\exists$ a live execution in which we

2. Fix $q_0, q_1, q_2, \ldots$, a live execution in which we never h

EI

3. $q_0, q_1, q_2, \ldots$ is either finite with $x = y$ in the final sta

Definition of "live execu

4. $q_0, q_1, q_2, \ldots$ is infinite By 3 and the fact

that we don't have x

5. $\exists$ a smallest value of $x + y$ that occurs in the states

Well-ordering

6. Fix q_k to be a state in which this smallest value is re

EI

7. The value of $x + y$ in q_{k+1} is smaller than the value

By the way the steps ar

8. q_k does not have the smallest value of $x + y$ in the e

By 7

9. QED Contradiction, 6 and 8.

Prove: In any live execution, there exists some state in which all the nodes are colored.

1. Fix $q_0, q_1, q_2, \ldots$, a live execution.
2. $q_0, q_1, q_2, \ldots$ is either finite with no step enabled in the final stat
   Definition of "live execution".
3. $q_0, q_1, q_2, \ldots$ isn't infinite — Number of uncolored nodes decrea every step, can't go below 0.
4. $q_0, q_1, q_2, \ldots$ is finite with no step enabled in the final state.
   By 2. and 3.
5. All nodes are colored in the final state.
   1. If not all nodes are colored in the final state, then some ste
      Any uncolored node can be colorec there are enough colors.
   2. QED — Contradiction (5.1. and 4.)
6. QED — EG, UG