Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professor Albert R. Meyer

Handout 62
May 20, 1999

# Notes for Final Exam

## Summation

$$\begin{aligned}
\sum_{i=0}^{n} x^i &= \frac{1-x^{n+1}}{1-x} \\
\sum_{i=0}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \ldots \\
\sum_{i=1}^{n} i x^i &= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2} \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
-\ln(1-x) &= \int \frac{1}{1-x} \;=\; x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots
\end{aligned}$$

## Master Theorem

For $T, f : \mathbb{N} \to \mathbb{N}$, and constants $a > 0, b > 1$, if

$$T(n) = aT(\lceil n/b \rceil) + f(n),$$

then

$$T(n) = \begin{cases} \Theta(n^\beta) & \text{if } f(n) = O(n^{\beta-\epsilon}) \text{ for some } \epsilon > 0, \\ \Theta(n^\beta \log n) & \text{if } f(n) = \Theta(n^\beta), \\ \Theta(f(n)) & \text{if } af(n/b) \le cf(n) \text{ for some } c < 1 \text{ and all large } n. \end{cases}$$

where $\beta ::= \log_b a$.

### Linear Recurrences

A homogeneous linear recurrence of order $d$,

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + \cdots + a_d f(n-d),$$

has characteristic polynomial:

$$p(\alpha) ::= \alpha^d - a_1\alpha^{d-1} - a_2\alpha^{d-2} - \ldots - a_{d-1}\alpha - a_d,$$

and characteristic equation $p(\alpha) = 0$. If $r$ is a root of $p$ with multiplicity $m > 0$, then all of

$$r^n,\ nr^n,\ n^2r^n,\ \ldots,\ n^{m-1}r^n$$

are solutions to the recurrence. Every solution to the equation is a linear combination of the $d$ terms obtained in this way from the characteristic equation.

### Stirling's Formula

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n e^{a_n}$$

where $1/(12n+1) \leq a_n \leq 1/12n$.

### Counting

Given a set of $n$ elements:

- number of $r$-permutations without replacement:

$$P(n,r) ::= n(n-1)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

- $r$-combination without replacement:

$$\binom{n}{r} ::= \frac{n!}{(n-r)!\ r!}$$

- $r$-permutation with replacement ($r$ distinct balls, $n$ distinct bins): $n^r$
- $r$-combination with replacement ($r$ identical balls, $n$ distinct bins):

$$\binom{n+r-1}{r}$$

- number of permutations with repetition:

$$\binom{n}{r_1, r_2, \ldots, r_k} ::= \frac{n!}{r_1!\ r_2! \ldots r_k!}$$

  where $r_1 + r_2 + \cdots + r_k = n$.

### Multinomial Theorem

$$(x_1 + x_2 + \ldots + x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \binom{n}{r_1, r_2, \ldots, r_k} x^{r_1} x^{r_2} \ldots x^{r_k}$$

### Inclusion-Exclusion

$$|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \left( \sum_{S \subseteq \{1,\ldots,n\}, |S|=k} |\bigcap_{i \in S} A_i| \right)$$

## Probability

### Events

$$\begin{aligned}
\Pr(\bigcup_{n\in\mathbb{N}} A_n) &= \sum_{n\in\mathbb{N}} \Pr(A_n) \text{ for pairwise disjoint } A_n \\
\Pr(\overline{B}) &= 1 - \Pr(B), \\
\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\
\Pr(A \cup B) &\leq \Pr(A) + \Pr(B) \quad \text{[Boole's inequality]} \\
\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\
\Pr(A \mid B) &= \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)} \quad \text{[Bayes Rule]}
\end{aligned}$$

### Expectation

$$\begin{aligned}
\mathrm{E}[R] &::= \sum_{r\in\mathbb{R}} r \cdot \Pr(R = r) \\
\mathrm{E}\left[\sum_{n\in\mathbb{N}} R_n\right] &= \sum_{n\in\mathbb{N}} \mathrm{E}[R_n] \quad \text{for convergent } \sum_{n=1}^{\infty} |R_n| \\
\mathrm{E}[R_1 R_2] &= \mathrm{E}[R_1]\,\mathrm{E}[R_2] \quad \text{for independent } R_1, R_2
\end{aligned}$$

**Variance**

$$\begin{aligned}
\mathrm{Var}[R] &::= \mathrm{E}\left[(R-\mathrm{E}[R])^2\right] = \mathrm{E}\left[R^2\right]-\mathrm{E}^2[R],\\
\sigma_R &::= \sqrt{\mathrm{Var}[R]} \quad [\sigma_R ::= \text{the } \textit{Standard Deviation} \text{ of } R],\\
\mathrm{Var}[aR+b] &= a^2\mathrm{Var}[R], \text{ for } a,b\in\mathbb{R},\\
\mathrm{Var}[R_1+\cdots+R_n] &= \mathrm{Var}[R_1]+\cdots+\mathrm{Var}[R_n] \quad \text{for pairwise independent } R_1,\ldots,R_n.
\end{aligned}$$

**Binomial Distribution**

$$\begin{aligned}
f_{n,p}(k) &::= \binom{n}{k}p^k(1-p)^{n-k}\\
F_{n,p}(k) &::= \sum_{i=0}^{k} f_{n,p}(i)\\
f_{n,p}(\alpha n) &= 2^{\left(\alpha\log_2\left(\frac{p}{\alpha}\right)+(1-\alpha)\log_2\left(\frac{1-p}{1-\alpha}\right)\right)\cdot n}\overbrace{e^{a_n-a_{\alpha n}-a_{n-\alpha n}}}^{\sim 1}\frac{1}{\sqrt{2\pi\alpha(1-\alpha)n}}\\
F_{n,p}(\alpha' n) &\le \frac{1-\alpha'}{1-\alpha'/p}f_{n,p}(\alpha' n)
\end{aligned}$$

where $0<\alpha<1, 0<\alpha'<p$, and $a_n$ from Stirling's formula.

**Wald's Theorem**

Let $Q$ be a non-negative integer valued random variable such that $\mathrm{E}[Q]$ is finite, and let $R_1, R_2, \ldots$ be a countable sequence of nonnegative random variables. Let $T$ be the sum $R_1+R_2+\cdots+R_Q$. Suppose that every variable $R_k$ has the same expectation, conditioned on being included in the sum, as $R_1$, and also suppose that $\mathrm{E}[R_1]$ is finite. Then, $\mathrm{E}[T]=\mathrm{E}[Q]\cdot\mathrm{E}[R_1]$.

**Markov's Theorem**

If $R: S\to\mathbb{R}^+$ is a random variable, then for all $x\ge 0$,

$$\Pr(R\ge x)\le\frac{\mathrm{E}[R]}{x}.$$

**The Chebychev Bound**

For any random variable R,

$$\begin{aligned}
\Pr(|R - \mathrm{E}[R]| \geq x) &\leq \frac{\mathrm{Var}[R]}{x^2}, \\
\Pr(|R - \mathrm{E}[R]| \geq x\sigma_R) &\leq \frac{1}{x^2}, \\
\Pr(R - \mathrm{E}[R] \geq x) &\leq \frac{\mathrm{Var}[R]}{\mathrm{Var}[R] + x^2}. \quad \text{[One-sided version]}
\end{aligned}$$

**Chernoff Bound**

Let $T$, be a sum of $n$ independent 0-1-valued random variables. Then for all $c > 1$

$$\begin{aligned}
\Pr(T \geq e\mathrm{E}[T]) &\leq e^{-\mathrm{E}[T]}, \\
\Pr(T \geq c\mathrm{E}[T]) &\leq e^{-\alpha c\mathrm{E}[T]},
\end{aligned}$$

where $e \approx 2.72$ is the base of the natural logarithms and $\alpha ::= \ln c + 1/c - 1$.