Mini-Quiz 3

1. Write your name:

2. **Velleman 2.3, Ex. 5:** Let $I = \{2, 3\}$, and for each $i \in I$ let $A_i = \{i, 2i\}$ and $B_i = \{i, i+1\}$.

   (a) List the elements of the sets A_i and B_i for all $i \in I$.

   (b) Find $\cap_{i \in I}(A_i \cup B_i)$ and $(\cap_{i \in I} A_i) \cup (\cap_{i \in I} B_i)$. Are they the same?

Tutorial 3 Problems

Problem 1 Well ordering

Rosen 3.2, Ex. 58: The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.
a) Show that S is nonempty.
b) Use the well-ordering property to show that S has a smallest element c.
c) Show that if d is a common divisor of a and b, then d is a divisor of c.
d) Show that $c|a$ and $c|b$. (Hint: First, assume that $c \not| a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.)
e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor of two positive integers is unique.

Problem 2 Induction and Recursive Algorithms

Procedure *binary search*(x, i, j)
$m := \lfloor (i + j)/2 \rfloor$
if $x = a_m$ **then**
  location $:= m$
else if ($x < a_m$ and $i < m$) **then**
  binary search$(x, i, m - 1)$
else if ($x > a_m$ and $j > m$) **then**
  binary search$(x, m + 1, j)$
else *location* $:= 0$

The above procedure searches x in a sequence $a_1, a_2, \ldots, a_n$ (when called with $i = 1$, $j = n$). Prove by (strong) induction that the binary search procedure sets *location* to an index i such that $a_i = x$, if there is such an index, and otherwise sets *location* to 0.

Problem 3 Relations

(a) Consider the following putative theorem and the accompanying proof.

Theorem. *For any sets A, B, C, and D, if $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$.*

Proof. Suppose $A \times B \subseteq C \times D$. Let a be an arbitrary element of A, b be an arbitrary element of B, and $(a, b) \in A \times B$. Since $A \times B \subseteq C \times D$, it follows that $(a, b) \in C \times D$. Therefore, $a \in C$ and $b \in D$. Since a and b were arbitrary elements of A and B, respectively, this shows that $A \subseteq C$ and $B \subseteq D$. ■

Is the proof correct? If so, what proof strategies does it use? If not, can it be fixed? Is the theorem correct?

(b) Let A be the adjacency matrix for a digraph G with n vertices. Note that A is a zero-one valued $n \times n$ matrix. In this problem we use the Boolean "OR" operation, $\vee$, instead of "+" when multiplying zero-one valued matrices (this is called "boolean" matrix product).

(i) For $k \in \mathbb{N}$, let A^k be the kth *Boolean* power of A. Prove that ${A^k}_{ij} = 1$ iff there is a directed path of length k from the ith to the jth vertex of G.

(ii) Let $B = A \vee I$ where I is the $n \times n$ identity matrix. Prove that $B^k_{ij} = 1$ iff there is a directed path of length $\leq k$ from the ith to the jth vertex of G.

(iii) Conclude that $B^n_{ij} = 1$ iff there is a directed path from the ith to the jth vertex.

Problem 4 Equivalence Relations Let $(a, b) \sim (c, d)$ if $a + d = b + c$. We claim that $\sim$ is an equivalence relation.

(a) Partition the set $\{0, 1, 2\} \times \{0, 1, 2\}$ into the distinct equivalence classes under $\sim$.

(b) Prove that $\sim$ is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$.

(c) How many distinct equivalence classes under $\sim$ are there on the set $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$. You do not need to prove your answer.