Tutorial 4 Problems

Problem 1 Scheduling Courses

We consider a simple kind of scheduling problem — for MIT courses with prerequisites. The problem is to determine how a student could complete all of the courses in the fewest terms, under the simplifying, if unlikely, assumption that there is no limit on the number of courses a student can take in a term as long as their prerequisites have all been met. We model this problem with a digraph. An example is shown in the figure below.

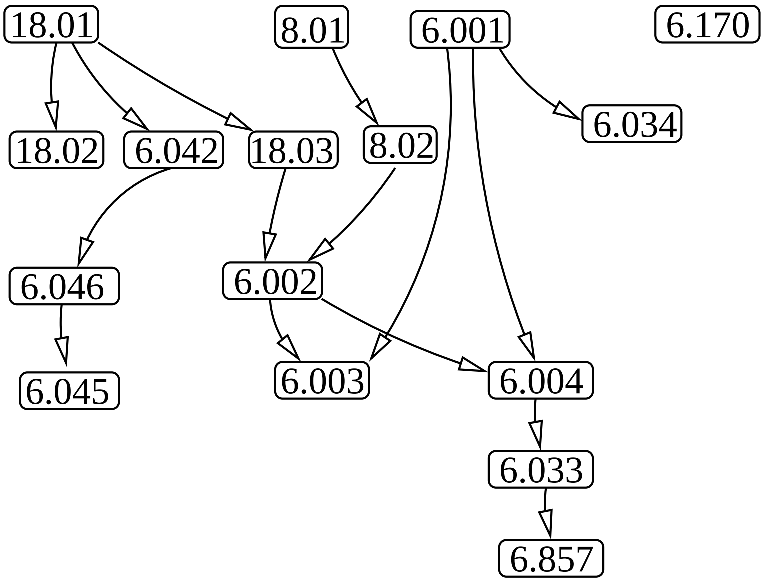


Figure 1: *This is a picture of a directed acyclic graph. Vertices are labeled with course names. An edge from u to v indicates that course u is a prerequisite for course v.*

(a) What's the length of a critical path? find such a path.

(b) How many terms are necessary and sufficient to complete all of the courses shown in the Figure, assuming prerequisites are observed, but course load is no issue? Prove your answer (both necessary and sufficient).

Solution:
The answer is suggested by the way the graph is drawn. Each term the student could take all the courses in a horizontal row, starting with the first row and working down. All of

the courses in each row are incomparable, and all the necessary prerequisites are in previous rows. Since there are six rows, the student can finish in six terms.

Furthermore, there is no way to finish in fewer than six terms, since each term the student can take at most one course in the six-course sequence 18.01, 18.03, 6.002, 6.004, 6.033, 6.857. Notice that this sequence constitutes the longest directed path in the graph.

(c) Show that there is a schedule for taking courses which achieves the minimum number of terms, t. the previous question includes this direction

Solution:
The solution to the previous part implies that t terms are sufficient. In term i, the student should take the set of courses in the ith block. Since the edges into any block come from an earlier block, all the prerequisites of courses in a block have been taken by the time the block is scheduled to be taken. And since the blocks partition the set of all courses, the student will complete every course.

Problem 2 Digraphs

Show that every tournament graph has a Hamiltonian path.

Solution:
We proceed by induction on n, the number of vertices in the tournament. The base case is $n = 2$, and the single edge is the Hamiltonian path. Now let G be a tournament with $n+1$ vertices. Delete one vertex, say v, and find (by the inductive hypothesis) a Hamiltonian path $v_1, v_2, \ldots, v_n$ in the tournament that remains. Now if $(v_n, v$ is anedge of G, then we have the Hamiltonian path $v_1, v_2, \ldots, v_n, v$. Similarly, if v, v_1 is an edge of G, then we have the Hamiltonian path $v, v_1, v_2, \ldots, v_n$. Otherwise, there must exist a smallest i such that (v_i, v) and (v, v_{i+1}) are edges of G. We can then splice v into the previous path to obtain the Hamiltonian path $v_1, v_2, \ldots, v_i, vi+1, \ldots, v_n$.

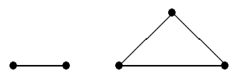
Problem 3 Let L_n be the $n+1$-vertex simple "line" graph consisting of a single simple path of length n. For example, L_4 is shown in the figure below:

The line graph L_4

Let's say a simple graph has "two ends" if it has exactly two vertices of degree one, and all its other vertices have degree two. In particular, for $n \geq 1$, the graph L_n has two ends.

(a) Draw a diagram of the smallest simple graph with two ends which is not isomorphic to any line graph. *Hint*: See part (d).

Solution:



The smallest two-ended graph not isomorphic to any line graph.

(b) Explain briefly, but clearly, what is wrong with the following argument:

False theorem: Every simple graph with two ends is isomorphic to L_n for some $n \geq 1$.

False proof: We prove by induction on the number, $n \geq 1$, of edges in a simple graph, that every two-ended graph with n edges is isomorphic to L_n.

(Base case $n = 1$): A simple graph with one edge can only consist of the two vertices connected by that edge and some number of vertices not attached to any edge, i.e., vertices of degree zero. Since a two-ended graph cannot have vertices of degree zero, the only two-ended graph with one edge must consist solely of the two vertices connected by the edge, which makes it isomorphic to L_1.

(Induction case): Assume that $n \geq 1$ and every two-ended graph with n edges is isomorphic to L_n. Now let G_n be any two-ended graph with $n \geq 1$ edges. By hypothesis, G_n is isomorphic to L_n. Suppose an edge is added to G_n to form a two-ended graph G_{n+1}.

Since G_n is isomorphic to L_n, it consists of a simple path of length n. The only way to add an edge to the path and preserve two-endedness is to have that edge go from one end—that is, one of the degree-one vertices—to a new vertex, lengthening the path by one. That is, the resulting $n+1$-edge graph must be a simple path of length $n+1$, so it is isomorphic to L_{n+1}. Q.E.D.

Solution:

The inductive case is argued in an incorrect way. The inductive case must consist of a proof for any $n \geq 1$ that if G_{n+1} is two-ended then $G_{n+1} \sim L_{n+1}$ ($A \sim B$ is a notation for "A is isomorphic to B"), using the assumption that for every two-ended G_n, $G_n \sim L_n$ (alternatively, in strong induction, we can use an assumption that for every k, $1 \geq k \geq n$, if G_k two-ended then $G_k \sim L_k$).

The false proof above does not argue that at all. Instead, it states that if one takes a two-ended G_n which is isomorphic to L_n, and extends it by an edge to form a two-ended $(n+1)$-edge graph G', then $G' \sim L_{n+1}$. This is true, but to transform this argument into an inductive proof, one would have to argue that every two-ended $(n+1)$-edge graph G_{n+1} can be formed in this way, i.e. by adding another edge to some two-ended graph G_n. However, this cannot be argued because it is not true. For example, in the solution to part (a), we have a two-ended graph G_4 which cannot be formed by adding an edge to some two-ended G_3.

(c) Here is another argument for the same False Theorem as in part (b).Explain exactly where it goes wrong.

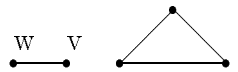
False proof 2: Same induction hypothesis and base case as in part (b).

(Induction case): For any $n \geq 1$, let G_{n+1} be any two-ended graph with $n+1$ edges. Let G_n be the graph which results from removing one of the degree-one vertices v of G_{n+1} and the edge $\{v, w\}$ attached to it. So G_n no longer has the vertex, v, of degree one. But the degree of w is one less in G_n than it was in G_{n+1}, so G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore G_n is also two-ended. By induction G_n consists of a simple path of length n. But G_{n+1} is obtained by attaching an edge from one end of the path to a vertex v not on the path, thereby lengthening the path by one. So G_{n+1} is isomorphic to a simple path of length $n+1$; that is, it is isomorphic to L_{n+1}. Q.E.D.

Solution:
The claim

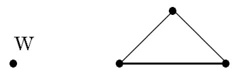
> G_n still has two vertices of degree one and one fewer vertex of degree two.

is wrong. The degree of w is one less in G_n than it was in G_{n+1}, hence it can be equal to one if it was two in G_{n+1}, but it can also be equal to zero if it was one in G_{n+1}. The false proof above is wrong because it ignores this this second possibility. For example, if we take $G_{n+1} = G_4$ from the solution to part (a):



Graph G_{n+1}

We see that if you remove v then w will have degree zero in a resulting graph G_n:



Graph G_n

Obviously, G_n is not a line graph.

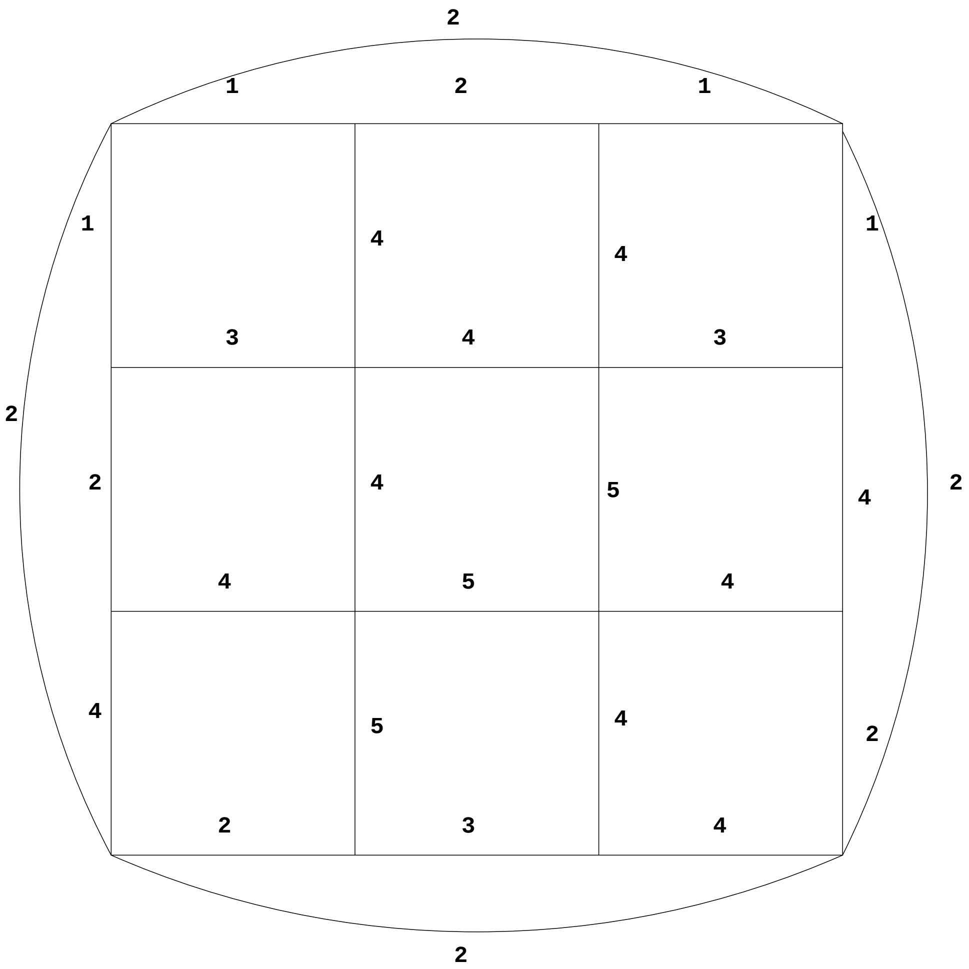
(d) Describe how to make a small revision to one of the false proofs above so that it becomes a correct proof of the theorem "Every *connected* simple graph with two ends is isomorphic to L_n for some $n \geq 1$."

Solution:
The proof in part (c) can be easily fixed if the graphs are connected, because then the quoted false claim will be true. Here is an example of how one could inject into the proof from part (c) an argument justifying the claim for the case of connected graphs:

> [...But the degree of w is one less in G_n than it was in G_{n+1}.] Since G_{n+1} was two-ended, w must be of degree either one or two in G_{n+1}. However, it cannot be of degree one because then vertices v and w would be connected only to each other and disconnected from the rest of the graph, which would contradict the assumption that G_{n+1} is connected. Therefore, w must have degree two in G_{n+1}, and hence has degree one in G_n. Hence, [G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore...]

The false proof in part (b) cannot be easily fixed. If we tried to use its structure to prove that all connected two-ended graphs are linear graphs, we would have to supply the argument that every connected two-ended graph can be formed by adding an edge to some connected two-ended graph with one less edge. To argue that would be as complicated as the above proof.

Problem 4 The diagram below depicts a weighted graph G, with integer weights on the edges as shown. Try to find a spanning tree of G with the smallest possible total edge weight (that is, a Minimum Spanning Tree).



Solution:

Note to the TA's: don't try to teach the general theory. Run a "competition". The students should devise various procedures for constructing a tree. Try to explain why this procedure gives the smallest possible tree.

I calculated the length of the minimum spanning tree to be 33. I could be wrong, though. Kruskal's Algorithm (Rosen, page 584) describes how to build a MST.