Massachusetts Institute of Technology
6.042J/18.062J: Mathematics for Computer Science
Professors Nancy Lynch and Srinivas Devadas

Handout 47
November 14, 2000

Problem Set 9 Solutions

2:35pm in lecture, Tuesday, November 21, 2000

Be sure to indicate your TA's name clearly on the solutions your turn in.

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Self-study Read: Rosen Sections 4.4 Self-study problems (not to be handed in): 19, 21, 28, 31, 33.

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Reading: Lecture notes; Rosen 4.4

Policy on collaboration If you collaborated with someone on the problem sets, you should mention their names on top of the problem set. If you did not collaborate with anyone, you should explicitly say so. As a reminder, the write-up of your solutions should be entirely your own.

Problem 1 Lets Make a Deal *differently*

(a) Suppose that you are the contestant on a variant of *Lets Make a Deal* where the keys to a new Corvette are behind one door, tickets for a dream vacation are behind a second door, and there is a diminutive, stuffed pig behind a third door. You do not know which door conceals which prize. Your objective is to maximize your probability of winning the pig.

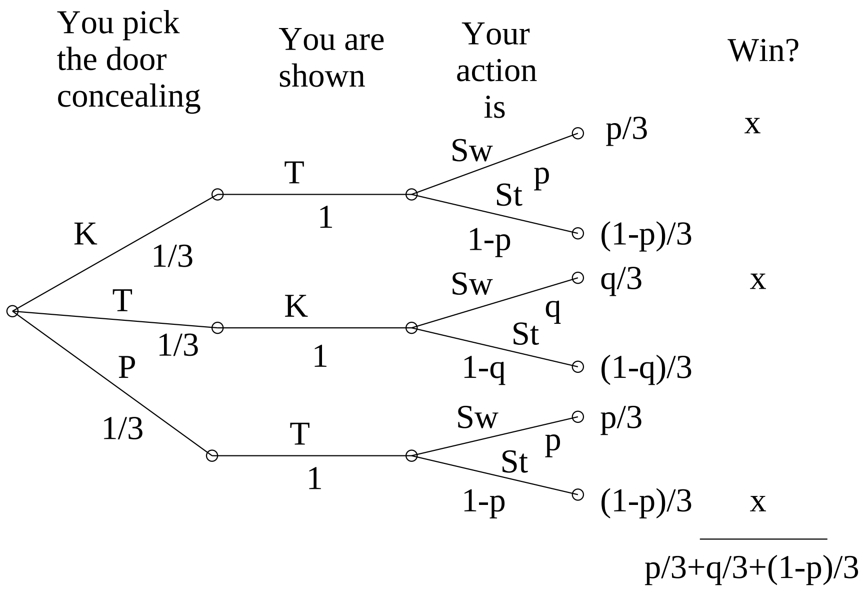
As before, you pick a door initially. Carol then opens a different door. Specifically, if you picked the door concealing the pig or the keys, then she reveals the tickets. If you picked the door concealing the tickets, then she reveals the keys. You can then either stay with your original door or switch to the remaining, closed door. You are awarded the prize behind the door that you choose at this second stage.

In substance, this variant differs from the original in that you know Carol's door-opening rules. This may give you a slight edge. Give a strategy that maximizes your probability of winning the pig and compute this probability.

Solution:
For the "stick" strategy, the probability of winning the pig is clearly $\frac{1}{3}$.

Let us now analyze the "switch" case. Let p be the probability that you switch if you are shown the tickets, and let q be the probability that you switch if you are shown the keys. The tree diagram shown below illustrates the various cases.

Your probability of winning is therefore



$$\frac{p}{3} + \frac{q}{3} + \frac{1-p}{3} \quad = \quad \frac{1}{3} + \frac{q}{3}.$$

This expression achieves its maximum value of $\frac{2}{3}$ when $q = 1$. This means that in an optimal strategy, you always switch if you are shown the keys. Your action when you are shown the tickets does not affect your chance of winning.

(b) Suppose that the gameshow is now modified so that there are four doors. As before, a prize is hidden behind one door. The contestant picks a door, Carol opens a different door to reveal no prize, and then the contestant is allowed to stay with his or her original door or to pick one of the other two that are still closed. If the contestant chooses the door concealing the prize in this second stage, then he or she wins it.

Contestant Stu, a sanitation engineer from Trenton, New Jersey, stays with his original door. What is the probability that he wins the prize?

Solution:
Stu guesses the right door initially with probability $\frac{1}{4}$ and guesses the wrong door with probability $\frac{3}{4}$. If he was right initially, then he surely wins the prize. If he was wrong initially, then he surely does not win the prize. Therefore, his probability of winning is $\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 0 = \frac{1}{4}$.

(c) Contestant Zelda, an alien abduction researcher from Helena, Montana, randomly picks one of the other two doors with equal probability. What is the probability that she wins the prize?

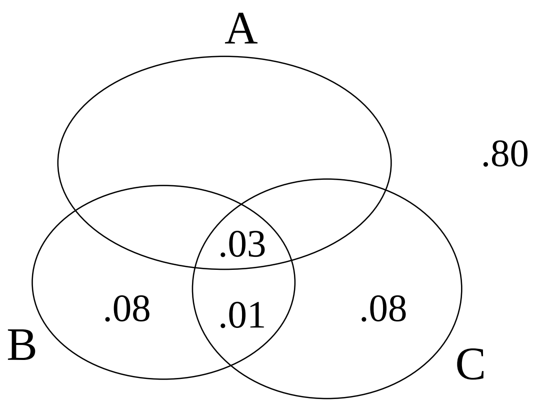
Solution:
Zelda also guesses the right door initially with probability $\frac{1}{4}$ and guesses the wrong door with probability $\frac{3}{4}$. If she was right initially, then she can not win. If she was wrong initially, then she wins with probability $\frac{1}{2}$. Therefore, her probability of winning is $\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$.

Problem 2 Probability Spaces In each problem part, either define a probability space and events A, B, and C such that all the equations hold simultaneously, or else prove that no such probability space and events exist.

(a)

$$\begin{aligned}\Pr(A \mid B) &= \frac{1}{4}\\ \Pr(A \mid C) &= \frac{1}{4}\\ \Pr(A \mid B \cap C) &= \frac{3}{4}\end{aligned}$$

Solution:
These events can exist, as indicated in the diagram below. Empty regions have zero probability.



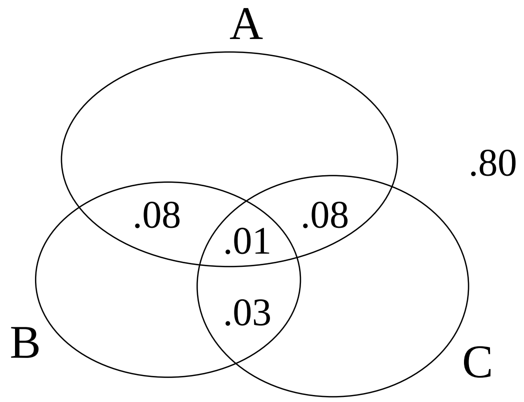
(b)

$$\begin{aligned}\Pr(A \mid B) &= \frac{3}{4}\\ \Pr(A \mid C) &= \frac{3}{4}\\ \Pr(A \mid B \cap C) &= \frac{1}{4}\end{aligned}$$

Solution:
These events can exist, as indicated in the diagram below.

(c)

$$\begin{aligned}\Pr(A \mid B) &= \frac{3}{4}\\ \Pr(A \mid C) &= \frac{3}{4}\\ \Pr(A \mid B \cup C) &= \frac{1}{4}\end{aligned}$$



Solution:
These events can not exist. At least one of the events B or C must be at least half as likely as $B \cup C$. Without loss of generality, assume that this is true for event B; that is,

$$\frac{\Pr(B)}{\Pr(B \cup C)} \geq \frac{1}{2}.$$

The first equation implies that

$$\frac{\Pr(A \cap B)}{\Pr(B)} = \frac{3}{4}.$$

Multiplying the equation and the inequality above gives

$$\frac{\Pr(A \cap B)}{\Pr(B \cup C)} \geq \frac{3}{8}.$$

Note that the set $A \cap (B \cup C)$ contains the set $A \cap B$. Therefore, $\Pr(A \cap (B \cup C)) \geq \Pr(A \cap B)$. Substituting this fact into the preceding inequality gives

$$\frac{\Pr(A \cap (B \cup C))}{\Pr(B \cup C)} \geq \frac{3}{8}.$$

Consequently, $\Pr(A \mid B \cup C) \geq \frac{3}{8} > \frac{1}{4}$, contradicting the third equation.

Problem 3 Suppose that I pick a pair of distinct numbers in the range 1 to 10 uniformly at random.

(a) If one of the numbers is even, then what is the probability that the other number is even?
Solution:
The sample space consists of the $\binom{10}{2} = 45$ pairs of distinct numbers in the range 1 to 10. Since the

sample space is uniform, each outcome has probability $\frac{1}{45}$. With this observation, we can compute the probability of an event by counting the number of outcomes in the event and then multiplying by $\frac{1}{45}$.

Let E be the event that at least one number is even, and let B be the event that both numbers are even. We must compute

$$\begin{aligned}\Pr(B \mid E) &= \frac{\Pr(B \cap E)}{\Pr(E)} \\ &= \frac{\Pr(B)}{\Pr(E)}\end{aligned}$$

The second equality holds because B is contained in E; that is, if both number are even, then at least one of the numbers is even.

Since there are 5 even numbers in the range 1 to 10, the event B contains $\binom{5}{2} = 10$ outcomes and thus has probability $10 \cdot \frac{1}{45} = \frac{2}{9}$. The event E consists of all pairs of numbers in the range 1 to 10 that are not both odd. That is, event E contains $45 - \binom{5}{2} = 35$ outcomes, and thus has probability $35 \cdot \frac{1}{45} = \frac{7}{9}$.

Substituting these probabilities into the above equation gives

$$\begin{aligned}\Pr(B \mid E) &= \frac{\Pr(B)}{\Pr(E)} \\ &= \frac{\frac{2}{9}}{\frac{7}{9}} \\ &= \frac{2}{7}.\end{aligned}$$

(b) If one of the numbers is a 2, then what is the probability that the other number is even?

Solution:

Let T be the event that one number is 2, and let U be the event that one number is 2 and the other is even. We must compute

$$\begin{aligned}\Pr(U \mid T) &= \frac{\Pr(U \cap T)}{\Pr(T)} \\ &= \frac{\Pr(U)}{\Pr(T)}\end{aligned}$$

The event T consists of 9 outcomes, since every pair must contain 2 and there are 9 choices for the other number $(1, 3, 4, \ldots, 10)$. The event U consists of 4 outcomes, since every pair must contain a 2 and there are 4 choices for the other number $(4, 6, 8, 10)$. Therefore, we have:

$$\begin{aligned}\Pr(U \mid T) &= \frac{\Pr(U)}{\Pr(T)} \\ &= \frac{\frac{4}{45}}{\frac{9}{45}} \\ &= \frac{4}{9}.\end{aligned}$$

Problem 4 **Tennis Tournaments.** A tennis tournament has 8 players. The players are assigned randomly to positions in the first round of a tournament ladder (see Figure 1).

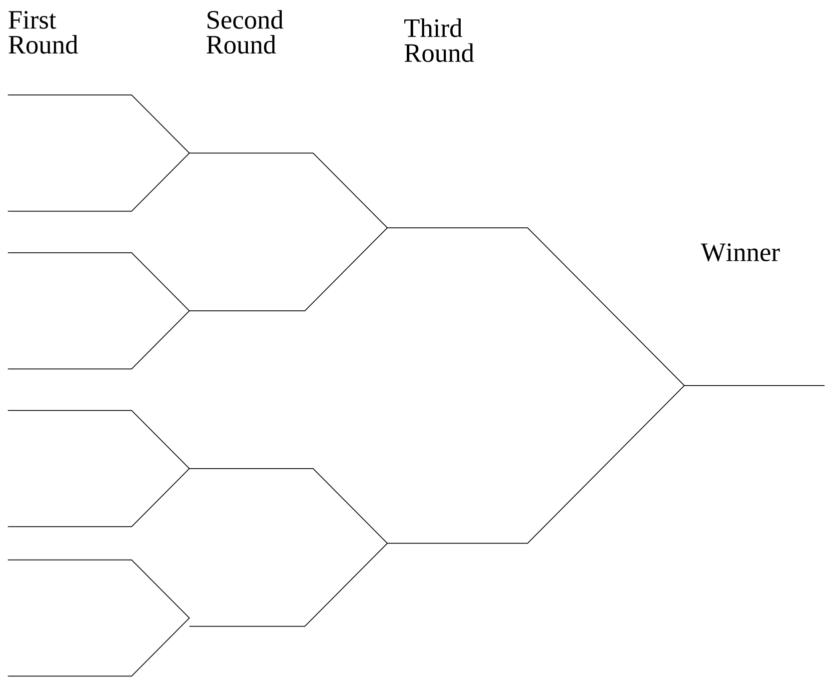


Figure 1.

(a) Suppose the *best player* always defeats everybody else, and the *second-best player* always defeats everybody but the best. What is the chance that the second-best player makes it to the final round?

Solution:
In order to make it to the final round, the second-best player has to meet the winner in the final round and not earlier. That happens if and only if the best player and the second-best player start out in different brackets of four; in other words, if the best player starts in position 1-4 of the first round, then the second-best must start in positions 5-8, and vice versa.

Here are two ways to determine the probability. The first way counts complete assignments: there are 8 ways to place the best player in the first round, then 4 ways to place the second-best, then 6! ways to place the remaining players. Divide by the size of the sample space (8!, the total number of ways to assign all eight players with no constraints) to get 4/7.

The second way treats the second player's position as a random selection. The best player can be placed anywhere. Given the best player's position, the second-best player will meet the first player in the final round only if he is placed in 4 of the remaining 7 positions—not 4 out of 8, because the best player is already occupying one (this is a common error – the positions of the players can't be treated like independent coin flips or dice rolls, because no two players can occupy the same position.)! Since all placements are equally likely, there is a 4/7 chance of this happening.

(b) Suppose the 8 tennis players are equally good, i.e., for any two players A and B,

$$\Pr\{A \text{ wins}\} = \Pr\{B \text{ wins}\} = 1/2,$$

and the twins Tom and Mot are amongst the 8 players. What is the chance that Tom and Mot ever meet in a match during the tournament?

Solution:
The probability that Tom and Mot will meet in the first round is 1/7 (Tom can be placed anywhere, but given Tom's position, Mot has only 1 choice in 7).

Tom and Mot will meet in the second round if and only if both win their first-round matches (with probability $1/2 \cdot 1/2$) and both are in different groups of two but the same group of four (probability 2/7).

Tom and Mot will meet in the third round if and only if both win two matches (with probability $1/2^2 \cdot 1/2^2$) and both are in different groups of four (probability 4/7).

Since these events are disjoint, the probability that Tom and Mot meet in any round is just the sum:

$$\frac{1}{7} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2}{7} + \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{4}$$

Problem 5 Consider the following procedure for generating a small, positive integer. I flip a coin. If the outcome is heads, then I roll one die and take the result. If the outcome is tails, then I roll two dice and take the sum. Assume that the coin is fair and the dice are fair and 6-sided.

(a) Define a probability space that describes this experiment.

The sample space consists of 42 outcomes. Six of these are pairs:

$$(H, 1), (H, 2), \ldots, (H, 6)$$

Each of these outcomes occurs with probability $\frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$. The remaining thirty-six outcomes are triples:

$$(T, 1, 1), (T, 1, 2), \ldots, (T, 6, 6)$$

Each of these outcomes occurs with probabiliy $\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{72}$.

(b) What is the probability that the procedure generates a 5?

$$\begin{aligned}
\Pr(\text{generates } 5) &= \Pr((H,5)) + \Pr((T,1,4)) + \Pr((T,2,3)) \\
&\quad + \Pr((T,3,2)) + \Pr((T,4,1)) \\
&= \frac{1}{12} + \frac{1}{72} + \frac{1}{72} + \frac{1}{72} + \frac{1}{72} \\
&= \frac{5}{36}
\end{aligned}$$

(c) What is the probability that the coin was heads given that the procedure generated a 5?

$$\begin{aligned}
\Pr(\text{coin was heads} \mid \text{generated } 5) &= \frac{\Pr(\text{coin was heads} \cap \text{generated } 5)}{\Pr(\text{generated } 5)} \\
&= \frac{\Pr((H,5))}{\frac{5}{36}} \\
&= \frac{\frac{1}{12}}{\frac{5}{36}} \\
&= \frac{3}{5}
\end{aligned}$$

Problem 6 Each person carries a certain, rare disease with probability 1 in 1000. Two doctors claim to be able to test for the disease before symptoms appear.

(a) Doctor A was educated at Harvard Medical School and employs the latest high-tech test. If a patient has the disease, then the test comes back positive 99% of the time. If the patient does not have the disease, then the test comes back negative 97% of the time. What is the probability that Doctor A correctly diagnoses a patient?

$$\begin{aligned}
\Pr(A \text{ is correct}) &= \Pr(\text{is diseased} \cap \text{test positive}) \\
&\quad + \Pr(\text{not diseased} \cap \text{test negative}) \\
&= \Pr(\text{is diseased}) \cdot \Pr(\text{test positive} \mid \text{is diseased}) \\
&\quad + \Pr(\text{not diseased}) \cdot \Pr(\text{test negative} \mid \text{not diseased}) \\
&= 0.001 \cdot 0.99 + 0.999 \cdot 0.97 \\
&\approx 0.970
\end{aligned}$$

(b) Doctor B (a quack from MIT) murmurs strange chants and burns incense. With probability 1 in 1000, Doctor B says that a patient has the disease, regardless of whether or not the patient actually does. What is the probability that Doctor B correctly disgnoses a patient?

$$\begin{aligned}
\Pr(B \text{ is correct}) &= \Pr(\text{is diseased} \cap \text{test positive}) \\
&\quad + \Pr(\text{not diseased} \cap \text{test negative}) \\
&= \Pr(\text{is diseased}) \cdot \Pr(\text{test positive} \mid \text{is diseased}) \\
&\quad + \Pr(\text{not diseased}) \cdot \Pr(\text{test negative} \mid \text{not diseased}) \\
&= 0.001 \cdot 0.001 + 0.999 \cdot 0.999 \\
&\approx 0.998
\end{aligned}$$

(c) Which doctor makes more errors? And how often, compared to the other?

Solution:
Doctor A makes about 15 times more errors than Doctor B.