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# Problem Set 4

**Due: 2:35pm in lecture, October 3, 2000.**

**Be sure to indicate your TA's name clearly on the solutions your turn in.**

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**Self-study: Matrices** Read: Rosen §2.6
Self-study problems (not to be handed in): Rosen §2.6, Exercises 1, 3, 5, 9, 13, 15, 23, 29, 31

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**Reading:** Rosen 7.1-7.5, 7.8, 8.1, 8.2, 8.4, 8.5

**New policy on collaboration** Starting from today: If you collaborated with someone on the problem sets, you should mention their names on top of the problem set. If you did not collaborate with anyone, you should explicitly say so. As a reminder, the write-up of your solutions should be entirely your own.

**Problem 1** At last count, our 6.042 class contains 180 students. Prove that the class must contain either: (a) A set $S$ of 15 students such that, if they line up in size places, they are also arranged in increasing order of age, or (b) a set $T$ of 13 students such that, if they line up in size places, they are also arranged in decreasing order of age.

**Solution:**
For each student, we can assign a pair $(s, a)$ where $s$ represents the size (height) of the student and $a$ represents the age of the student. We then get a set $U$ of 180 such pairs. We can define the following binary relation $R$, denoted by $\preceq$, between two pairs $P_1$ and $P_2$: $P_1 \preceq P_2$ iff $s_1 \leq s_2$ and $a_1 \leq a_2$.

For example, if $P_1 = (180, 20)$, $P_2 = (180, 21)$, $P_3 = (185, 19)$, we have $P_1 \preceq P_2$ but $P_3$ and $P_1$ are incomparable.

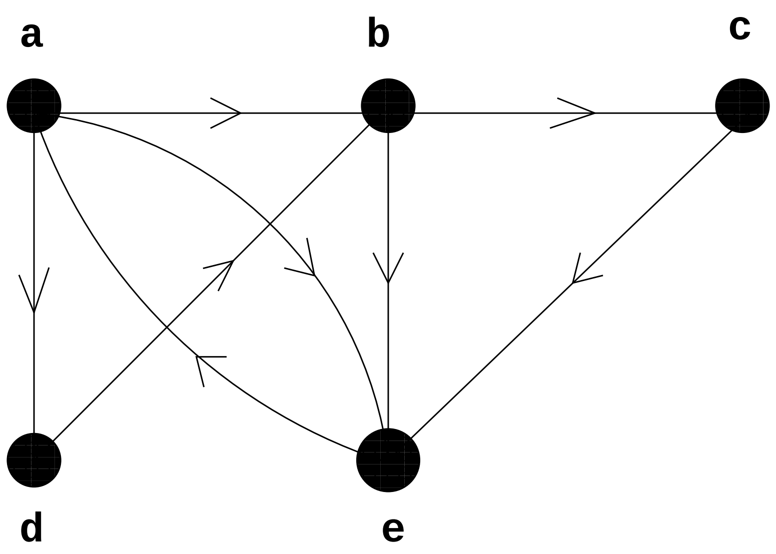
It is easy to check that $R$ is reflexive, transitive and antisymmetric. Thus, it defines a partial order on the set $U$. By Dilworth's Theorem, since $U$ has 180 elements, it must either have a chain of size$> 14$ or an antichain of size $\geq \frac{180}{14}$.

**Case 1** $U$ has a chain of size $> 14$. Then it has a chain of size at least 15. This means that we can find 15 pairs which satisfy the relation $P_1 \preceq P_2 \preceq P_3 \ldots \preceq P_{15}$. From the definition of our binary relation ($\preceq$), this implies that $s_1 \leq s_2 \leq s_3 \leq \ldots \leq s_{15}$ AND $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_{15}$. Therefore, if we place the corresponding 15 students in the same order, they will be lined up in in increasing order of size AND in increasing order of age.

**Case 2** $U$ has an antichain of size $\geq \frac{180}{14}$, so, since antichains only come in integer lengths, it has an antichain of size at least 13. Therefore, we can find 13 pairs which are incomparable. We can line them up in order of increasing size (ignoring age), and come up with an order $P_1 \ldots P_{13}$ so that $s_i \leq s_{i+1} (i = 1 \ldots 12)$. Since for all $i$, $s_i \leq s_{i+1}$ and $P_i, P_{i+1}$ are incomparable, we can conclude that $a_i > a_{i+1}$ (because if not, $a_i \leq a_{i+1}$ so for that $i$, $s_i \leq s_{i+1}$, $a_i \leq a_{i+1}$, therefore $P_i \preceq P_{i+1}$, so $P_i$ and $P_{i+1}$ would NOT be incomparable.) If we take the students corresponding to $P_1 \ldots P_{13}$ and line them up in that order, they will be in increasing order of size (since $s_i \leq s_{i+1}$) and in decreasing order of age (since we established that $a_i > a_{i+1}$).

To eliminate some confusion: the sequence $\{2, 2, 3, 4\}$ is in increasing order, but not in strictly increasing order.

**Problem 2** Find the number of paths of length 5 from a to e of in the directed graph shown below




**Solution:**

(a) a-b b-e e-a a-b b-e

(b) a-b b-c c-e e-a a-e

(c) a-e e-a a-e e-a a-e

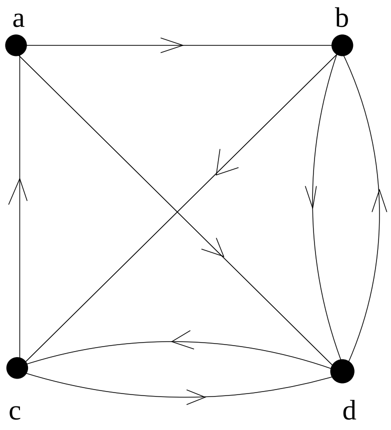
(d) a-e e-a a-b b-c c-e

(e) a-e e-a a-d d-b b-e

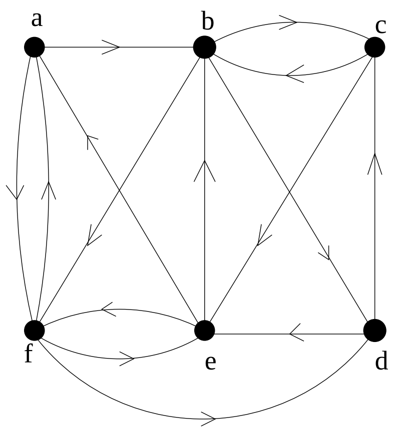
(f) a-d d-b b-e e-a a-c

**Problem 3** An *Euler path* in a digraph $G$ is a path that traverses all the edges of $G$, each edge exactly once. Unlike an Euler circuit, it need not end up at the same vertex where it started.

Say whether each of the following digraphs has an Euler path, an Euler circuit, both, or neither. Justify your answers by exhibiting the circuits and paths that you say exist, and proving that the others don't exist.




(a)




(b)

**Solution:**

(a) **Euler Circuit** NO, because vertex $a$ has in-degree 1 and out-degree 2

**Euler Path** YES. Since it doesn't have an Euler Circuit and exactly 2 vertices have odd degree (i.e. $a$, $d$). E.g: $a, b, d, b, c, d, c, a, d$

(b) **Euler Circuit** YES. E.g.: $a, b, c, e, b, d, c, b, f, d, e, f, e, a, f, a$

**Euler Path** YES. If a graph has an Euler Circuit, then it also has an Euler path.

**Problem 4** For each of the following graphs,

- Find the chromatic number

- Say whether it is possible to decrease the chromatic number by removing a single vertex and all edges incident with it.

Justify your answers.

**(a)** The graph in Rosen Section 7.8, exercise 4.

**(b)** The graph in Rosen Section 7.8, exercise 8.

**Solution:**

(a) Since there is a triangle, at least 3 colors are needed. To show that 3 colors suffice, notice that we can color the vertices around the outside alternately using red and blue, and color vertex $g$ green.

If we remove vertex $g$, the chromatic number will be reduced to 2. We can color vertices $f, b, d$ blue and $a, c$ and $e$ red

Even if we remove a vertex, at least one of the two triangles $ace$ and $bdf$ must remain since they share no vertices. Therefore, the smaller graph will still have chromatic number 3.

(b) Since vertices $b, c, h, i$ form a $K_4$, at least 4 colors are needed. A coloring using only 4 colors is indeed possible: eg: let $a$ and $c$ be red, $b, d$ and $f$ blue,: $g$ and $i$ green and $e$ and $h$ yellow.

If we remove vertex $h$ and recolor vertex $e$ red, then we can eliminate color yellow from the solution, thus reducing the chromatic number to 3.

**Problem 5** Rosen section 7.5, exercise 44.

**Solution:**

This graph has no Hamiltonian circuit. If it did, then certainly the circuit would have to contain edges $d, a$ and $a, b$, since these are the only edges incident to vertex $a$. By the same reasoning, the circuit would have to contain the other six edges around the outside of the figure. These eight edges already complete a circuit, and this circuit omits the nine vertices on the inside. Therefore there is no Hamiltonian circuit.

**Problem 6** Either describe what a full $m$-ary tree with 76 leaves and height 3, where $m$ is a positive integer, would look like, or show that no such tree exists.

**Solution:**

Such a tree does exist. We first note that such a tree must have $i = \frac{75}{m-1}$ internal vertices. This has to be a whole number, so $m-1$ must divide 75. This is possible, for example when $m = 6$. Let's try to construct such a tree: A complete 6-ary tree of height 2 would have 36 leaves. We therefore need to add 40 leaves. This can be accomplished by changing 8 vertices at level 2 to internal vertices; each such change adds 5 leaves to the tree (6 new leaves at level 3, minus the one leaf at level 2 that has been changed to an internal vertex). So the tree looks like this: The root has 6 children, each of which has 6 children, giving 36 vertices at level 2. Of these, 28 are leaves and each of the remaining 8 vertices at level 2 has 6 children living at level 3, for a total of 48 leaves at level 3. The total number of leaves is thus $48 + 28 = 76$, as desired.

**Problem 7** Rosen 8.2 exercise 8. How many weighings of a balanced scale are needed in order to find a counterfeit coin among eight coins if the the counterfeit coin is either heavier or lighter than the others? Describe an algorithm to find the counterfeit coin using this number of weighings.

**Solution:**

Decision tree theory applied naively says that at least $log_3 8 = 2$ weighings are needed, but in fact at least 3 weighings are needed. To see this, consider what the first weighing might accomplish. We can put one, two or three coins in each pan. If we put one in each pan, and the scale balances, what we know is that the counterfeit must be among the remaining 6. Similarly, if we put 2 and the scale balances, we narrow down our list of suspects to the remaining 4. If we put 3 in each pan and the scale balances, we are in really good shape, but what if it doesn't? Then our list of suspects will be the 6 coins in the scale. What this means is that however many coins we choose to put on each pan (1,2 or 3), there's always a case where the list of suspects after the first weighing will be $\geq 4$. One weighing is not enough to distinguish the counterfeit from among a list of 4 suspects.

In Tutorial 1, we showed that we can solve a bigger problem (12 coins) with three weighings, so we should be able to solve this one with the same number of weighings. So if we are lazy and we want to just use the algorithm we came up with in Tutorial, we just add 4 good coins to our collection and apply that algorithm.

However, a simpler solution is available: Put two coins in each pan. If the scale balances, then the search is reduced to the other four coins. If the scale does not balance, then the counterfeit is among the four coins on the scale. In both cases, after 1 weighing we have cut down our list of suspects to 4. Call them A, B, C, D. First compare coins A and B. If they balance, then the counterfeit must be among the other 2. In this case, compare C with A (which we now know must be good). If they balance, then D is counterfeit. If not, C is counterfeit. On the other hand, if A and B do not balance, then one of them is the counterfeit. Compare A with C (which we know must be good). If they balance, B is the counterfeit. If they do not, A is the counterfeit.