Problem Set 10

Due: 2:35pm in lecture, Tuesday, November 28, 2000.

Be sure to indicate your TA's name clearly on the solutions your turn in.

Self-study Read: Rosen Sections 4.4 Self-study problems (not to be handed in): 13, x, x, x.

Reading: Lecture notes; Rosen 4.4, 4.5

Policy on collaboration If you collaborated with someone on the problem sets, you should mention their names on top of the problem set. If you did not collaborate with anyone, you should explicitly say so. As a reminder, the write-up of your solutions should be entirely your own.

Problem 1 Conditional Probability

Suppose that 5 men out of 100, and 25 women out of 10,000 are colorblind. A colorblind person is chosen at random from a population with equal numbers of males and females. What is the probability of this person being male?

Solution: Let M be the event the person is male, F female, C colorblind. We are given $\Pr\{C \mid M\} = \frac{5}{100}$ and $\Pr\{C \mid F\} = \frac{25}{10{,}000}$. Since the population consists of equal numbers of males and females, $\Pr\{M\} = \Pr\{F\} = \frac{1}{2}$. So,

$$\Pr\{M \mid C\} = \frac{\Pr\{C \mid M\} \times \Pr\{M\}}{\Pr\{C\}} = \frac{\Pr\{C \mid M\} \times \Pr\{M\}}{\Pr\{C \mid M\} \times \Pr\{M\} + \Pr\{C \mid F\} \times \Pr\{F\}} =$$

$$\frac{(5/100) \times (1/2)}{(5/100) \times (1/2) + (25/10000) \times (1/2)} = 20/21.$$

Problem 2 Suppose that a resident of Pittsburgh chosen at random is a Steelers fan with probability 0.90, that a resident chosen at random is a Penguins fan with probability 0.75, and that a resident Steelers fan chosen at random is a Penguins fan with probability 0.80. Now suppose that a resident is chosen at random from among those that are *not* Steelers fans. What is the probability that such a resident is a Penguins fan?

Solution: Let P represent the event that a resident is a Penguins fan, and S represent the event that the resident is a Steelers fan. Then:

$$\begin{aligned}
\Pr\{P\} &= \Pr\{P \mid S\}\Pr\{S\} + \Pr\{P \mid \overline{S}\}\Pr\{\overline{S}\} \\
\Pr\{P \mid \overline{S}\} &= \frac{\Pr\{P\} - \Pr\{P \mid S\} * \Pr\{S\}}{\Pr\{\overline{S}\}} \\
&= \frac{0.75 - 0.80 \cdot 0.90}{1 - 0.9} \\
&= 0.3
\end{aligned}$$

Problem 3 A population consists of c_1 members of type 1, c_2 members of type $2, \cdots, c_n$ members of type n. A set of k members is drawn from the population without replacement.

(a) Find the probability that i members of type 1 were chosen.

Solution: Let N_{1i} be the event that i members of class 1 were chosen. $\Pr\{N_{1i}\} = \frac{\binom{c_1}{i} \times \binom{c_2+\cdots+c_n}{k-i}}{\binom{c_1+\cdots+c_n}{k}}$.

(b) Find the probability that i members of class 1 and j members of class 2 were chosen.

Solution: Let N_{2j} be the event that j members of class 2 were chosen. Then $\Pr\{N_{1i} \cap N_{2j}\} = \frac{\binom{c_1}{i}\binom{c_2}{j}\binom{c_3+\cdots+c_n}{k-i-j}}{\binom{c_1+\cdots+c_n}{k}}$

Problem 4 In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time – i.e. the probability that it rains, given that she has predicted that it will rain, and the probability that it does not rain given that she has predicted that it does not rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3. Find:
a) the probability that Pickwick has no umbrella, given that it rains.
b) the probability that it doesn't rain, given that he brings his umbrella.

Solution:

We construct a tree as in lecture notes #19. For part (a), we wish to find

$$\begin{aligned}
\Pr\{\text{no umbrella} \mid \text{rain}\} &= \frac{\Pr\{\text{no umbrella} \wedge \text{rain}\}}{\Pr\{\text{rain}\}} \\
&= \frac{1/9}{1/2} = 2/9.
\end{aligned}$$

For part (b), we wish to find

$$\begin{aligned}
\Pr\{\text{no rain} \mid \text{umbrella}\} &= \frac{\Pr\{\text{no rain} \wedge \text{umbrella}\}}{\Pr\{\text{umbrella}\}} \\
&= \frac{1/6 + 1/9}{1/9 + 1/6 + 1/3 + 1/18} = 5/12
\end{aligned}$$

Problem 5

(a) Suppose that you are looking in your desk for a letter from some time ago. Your desk has eight drawers, and you assess the probability that it is in any particular drawer as 10% (so there is a 20% chance that it is not in the desk at all). Suppose now that you start searching systematically through your desk, one drawer at a time. In addition, suppose that you have not found the letter in the first i drawers, where $0 \leq i \leq 7$. Let p_i denote the probability that the letter will be found in the next drawer, and let q_i denote the probability that the letter will be found in some subsequent drawer (both p_i and q_i are conditional probabilities, since they are based on the assumption that the letter is not in the first i drawers). Show that the p_i's increase and the q_i's decrease.

Hint: Observe that if event A implies event B then $\Pr\{A \mid B\} = \Pr\{A\} / \Pr\{B\}$.

Solution:

First, we verify the hint. Saying that A implies B is the same as saying that that $A \cap B = A$, so

$$\Pr\{A \mid B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}} = \frac{\Pr\{A\}}{\Pr\{B\}}.$$

Now, we define the following events:

$$\begin{aligned} E_i &= \text{letter is in drawer } i, \\ N_i &= \text{letter is not in drawers } 1, 2, \ldots i, \\ R_i &= \text{letter is in one of drawers } i+1, i+2, \ldots 8. \end{aligned}$$

Now we wish to find

$$\begin{aligned} p_i &= \Pr\{E_{i+1} \mid N_i\} \\ &= \frac{\Pr\{E_{i+1} \cap N_i\}}{\Pr\{N_i\}} \\ &= \frac{\Pr\{E_{i+1}\}}{\Pr\{N_i\}} \qquad \text{(from the hint, since } E_{i+1} \Rightarrow N_i) \\ &= \frac{0.1}{0.2 + 0.1(8-i)} = \frac{1}{10-i}, \end{aligned}$$

which clearly increases as i increases, since the denominator gets smaller while the numerator remains constant.

Now,

$$\begin{aligned} q_i &= \Pr\{R_i \mid N_i\} \\ &= \frac{\Pr\{R_i \cap N_i\}}{\Pr\{N_i\}} \\ &= \frac{\Pr\{R_i\}}{\Pr\{N_i\}} \qquad \text{(from the hint, since } R_i \Rightarrow N_i) \\ &= \frac{0.1(8-i)}{0.2 + 0.1(8-i)} \\ &= \frac{8-i}{10-i} = \frac{10-i}{10-i} - \frac{2}{10-i} = 1 - \frac{2}{10-i} \end{aligned}$$

which decreases as i increases, since the denominator of the fraction shrinks, so the fraction becomes greater, so the negation of the fraction becomes lesser.

Problem 6 (12 points)

Let A, B, C be events of some experiment. For each of the following statements, prove it if it is true or give a counterexample if it is false:

(a) If C is independent of A and C is independent of B then C is independent of $A \cup B$.

Solution: The statement is false. For example let A and B be unbiased independent bits, and let $C = A \oplus B$. Then C is independent from both A and B, but it is not independent from $A \cup B = A \,\textsf{or}\, B$

(b) If C is independent of A, and C is independent of B, and C is independent of $A \cap B$ then C is independent of $A \cup B$

Solution: It is true. To prove it, observe by the Sum Rule that

$$\begin{aligned}
&\Pr\{x \in (C \cap (A \cup B))\} \\
&\quad = \Pr\{x \in (C \cap (A - B))\} + \Pr\{x \in (C \cap (B - A))\} + \Pr\{x \in (C \cap (A \cap B))\},
\end{aligned}$$

and then that

$$\Pr\{x \in (C \cap (A - B))\} + \Pr\{x \in (C \cap (A \cap B))\} = \Pr\{x \in (C \cap A)\}$$

also by the Sum rule. Then use the fact that C is independent of both A and $A \cap B$ to get that C is independent from $A - B$. And similarly that it is independent from $B - A$.

Problem 7 (15 points) Compute the probabilities of the following three events and determine which one is most likely. Assume that the dice are fair, 6-sided, and mutually independent.

(a) Rolling at least one 6 when six dice are rolled.

Solution: The probability of rolling a 6 on a particular die is $\frac{1}{6}$. The probability of not rolling a 6 on a particular die is $\frac{5}{6}$, since this is the complementary event. The probability of not rolling a 6 on any of six dice is $(\frac{5}{6})^6$, since the numbers rolled are mutually independent. The probability of rolling a 6 on some die is therefore $1 - (\frac{5}{6})^6 = 0.665\ldots$.

(b) Rolling at least two 6's when twelve dice are rolled.

Solution: The probability of rolling two or more 6's is 1 minus the probability of rolling zero or one 6's. The probability of rolling zero 6's is $(\frac{5}{6})^{12}$, arguing as before. The probablity of rolling exactly one 6 is

$$12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11}$$

since there are 12 ways to choose the die that comes up 6, the probability of this die coming up 6 is $\frac{1}{6}$, and the probability of the remaining dice all coming up in the range 1 to 5 is $(\frac{5}{6})^{11}$.

Overall, the probability of rolling two or more sixes is therefore

$$1 - \left(\frac{5}{6}\right)^{12} - 12 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{11} = 0.618\ldots.$$

(c) Rolling at least three 6's when eighteen dice are rolled.

Solution: Arguing as before, the probability of rolling at least three 6's is

$$1 - \left(\frac{5}{6}\right)^{18} - 18 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^{17} - \binom{18}{2} \cdot \left(\frac{1}{6}\right)^{2} \cdot \left(\frac{5}{6}\right)^{16} = 0.597\ldots.$$

Solution: The first event is most likely.

Problem 8 (12 points)

Given a biased coin that comes up heads 2/3 of the time and tails 1/3 of the time, simulate a fair coin by applying repeated flips of the unfair coin. In other words, design a procedure involving tosses of the unfair coin which has two outputs, "tail" and "head", and which outputs them with equal probability. Your procedure should return "head" or "tail" with probability at least $1 - (\frac{1}{2})^n$, using $O(n)$ tosses of the unfair coin, which means that your procedure can have a tiny probability of not returning any answer after using reasonably many tosses of the unfair coin.

Solution: The procedure is this: Flip the unfair coin twice. If you get HT then output "tail" and if TH then output "head". Otherwise repeat. Probability of outputting "tail" is equal to probability of outputting "head". Furthermore, probability of failure to output anything at all after using $2k$ tosses is $(\frac{2}{3} * \frac{2}{3} + \frac{1}{3} * \frac{1}{3})^k = (\frac{5}{9})^k = (\frac{1}{2})^{ck}$ for some small $c = (\ln \frac{5}{9} / \ln \frac{1}{2}) \approx 0.85$.

Problem 9 (15 points)

A *fair bit* is a random variable taking only the values zero and one, each with probability 1/2. It turns out that from a few *mutually* independent fair bits, we can create many more *pair-wise* independent fair bits. However we cannot create a larger set of mutually independent fair bits.

(a) Prove that two fair bits X and Y are independent if and only if $X \oplus Y$ (xor of X and Y) is a fair bit (xor of two bits is 0 if the bits are the same and 1 if they are different.)

Solution: "only if" direction is immediate. "if" direction is true because if xor is unbiased than probability of that (X, Y) is $(0, 0)$ or $(1, 1)$ is $1/2$. And because X is unbiased, this means that probability of each one is $1/4$. Similarly, you get that probability of both $(0, 1)$ and $(1, 0)$ is $1/4$, and hence X and Y are independent.

(b) Consider three mutually independent fair bits X, Y, and Z, and consider the following seven random variables: X, Y, Z, $X \oplus Y$, $X \oplus Z$, $Y \oplus Z$, $X \oplus Y \oplus Z$. Argue your answers to the following questions:

1. Are these variables fair bits?
2. Are they pair-wise independent?
3. Are they mutually independent?

Solution: yes,yes,no: all easy to check

(c) Now, given n mutually independent fair bits $X_1, ..., X_n$, construct $(2^n - 1)$ pair-wise independent fair bits, and prove that your construction works.

Solution: For every binary string $b \in \{0,1\}^n \neq (00...0)$, define $Y_{b_1 b_2 ... b_n} = \oplus_{b_i=1} X_i$. It is easy to argue that each Y is a random unbiased bit, and then that for every $b \neq b'$, $Y_b \oplus Y'_b$ is also an unbiased random bit, and hence Y_b and Y'_b are independent.

(d) Prove that there is no construction which given n mutually independent fair bits can create $n+1$ mutually independent fair bits. In other words, show that there is no function f mapping n mutually independent fair bits to $n+1$ mutually independent fair bits.
Hint: Show it first for $n = 2$ and then try to generalize your argument.

Solution: Assume for the purpose of contradiction that such a function exists. The event that all input bits are equal to zero occurs with probability $\frac{1}{2^n}$. In this event, the $n+1$ output bits take on some values. This implies that the $n+1$ output bits take on these values with probability at least $\frac{1}{2^n}$ and not $\frac{1}{2^{n+1}}$, which contradicts the assumption that the $n+1$ output bits are unbiased and mutually independent.

Question for the curious (not graded): Can you create 2^n pair-wise independent fair bits from n mutually independent fair bits? Can you see the answer for $n = 2$?

Solution: For $n = 2$, a bit variable derived from two bits is a function from $\{0,1\} \times \{0,1\}$ to $\{0,1\}$. There are $2^4 = 16$ such functions. Now, only 6, i.e. (4 choose 2) of them give an unbiased bit. In fact we can list them. If the inputs bits are X and Y then the 6 possible "derived" random bits are the following:

$$X, Y, \neg X, \neg Y, X \oplus Y, \neg(X \oplus Y)$$

You can quickly see that no four of these can be mutually independent.

However, I don't see how to generalize this fact to all n, or even to $n = 3$.