Massachusetts Institute of Technology — Handout ??
6.042J/18.062J: Mathematics for Computer Science — December 8, 2000
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Tutorial 12 Problems

Problem 1 In lecture, we computed upper and lower bounds on the probability that at least one event out of a set $A_1, \ldots, A_N$ occurs. The upper bound was:

Fact 1. *Let $A_1, A_2, \ldots, A_N$ be events over the same sample space, and let the random variable T be the number of these events that occur. Then*

$$\Pr(T \geq 1) \quad \leq \quad \sum_{i=1}^{N} \Pr(A_i).$$

The lower bound was:

Fact 2. *Let $A_1, A_2, \ldots, A_N$ be events over the same sample space, and let the random variable T be the number of these events that occur. Then*

$$\Pr(T \geq 1) \geq \max_{1 \leq i \leq N} \Pr(A_i).$$

Now assume that the events A_i are mutually independent. In this case, we can obtain a much better upper bound on the probability no events occur:

Theorem 0.1. *Let $A_1, A_2, \ldots A_N$ be independent events, and let T be the number of these events that occur. The probability that none of the events occur is at most $e^{-Ex(T)}$.*

(a) Prove this.

Hint: The proof of the theorem is surprisingly simple. The main trick is to use the fact that $1 - x \leq e^{-x}$ for all x. This inequality follows from the Taylor expansion $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$.

(b) A Mind-Reading Trick

For this trick, every card has a "value". The value of a number card less than 10 is just the number, and the value of a 10, face card, ace, or joker is 1. For example, $8\clubsuit$ has value 8, $Q\spadesuit$ has value 1, and $10\diamondsuit$ has value 1.

A volunteer shuffles the deck and secretly chooses a number from 1 to 9. She deals this number of cards minus one from the top of the deck and looks at the new top card. For example, if she chooses the number 4, then she deals off 3 cards and looks at the 4th card.

She then takes the value of this top card, deals off that number of cards minus one, and again looks at the new top card. For example if she sees a $6\diamondsuit$, then she deals off 5 more cards and looks at the next one.

She repeats this procedure until she runs out of cards. "Her card" is the last card that she looked at to determine the number to deal off. For example, suppose that the deck is as follows:

$$Q\heartsuit \; 4\spadesuit \; \underline{2\spadesuit} \; J\clubsuit \; \underline{K\spadesuit} \; \underline{5\heartsuit} \; 3\heartsuit \; A\clubsuit \; 7\diamondsuit \; 7\heartsuit \; \underline{3\clubsuit} \; K\clubsuit$$

If the volunteer chooses 3 as her secret number, then she deals off $3-1=2$ cards and looks at the $2\spadesuit$. Since the value of this card is 2, she deals off $2-1=1$ more card and looks at the $K\spadesuit$. This card has value 1, so she deals off $1-1=0$ more cards and looks at the $5\heartsuit$. Since this card has value 5, she deals of $5-1=4$ more cards and looks at the $3\clubsuit$. This card has value 3, so she tries to deal off $3-1=2$ more cards. However, she then runs out of cards, so "her card" is $3\clubsuit$.

After the volunteer completes the procedure, the magician is able to name her card!

The secret is that the magician performs the same procedure as the volunteer, starting with the number 1. When he guesses "her card", he is actually naming "his card"! Remarkably, the two are almost certain to be the same! The reason is that if the magician and volunteer ever land on the same card, then they will finish with the same card. In the example, the magician lands on the underlined sequence of cards:

$$\underline{Q\heartsuit}\ \underline{4\spadesuit}\ 2\spadesuit\ J\clubsuit\ K\spadesuit\ \underline{5\heartsuit}\ 3\heartsuit\ A\clubsuit\ 7\diamondsuit\ 7\heartsuit\ \underline{3\clubsuit}\ K\clubsuit$$

Even though the volunteer and the magician start with different numbers, they both coincidentally land on the $5\heartsuit$. After this point, they both continue in the same way and both end up with $3\clubsuit$.

All that remains is to prove that the magician and the volunteer are likely to land on the same card at some point. An exact analysis of this probability is difficult, but Theorem 0.1 gives some intuition.

Let A_i be the event that on his i-th jump the magician lands on a card also visited by the volunteer. If the volunteer's next card is 1 away, then the magician has a $\frac{1}{3}$ chance of landing on it (12 face cards, 3 10's, and 2 jokers gives $\frac{18}{54}=\frac{1}{3}$). If the volunteer's next card is more than 1 away, then the magician has a $\frac{2}{27}$ chance of landing on it (4 correct number cards gives $\frac{4}{54}=\frac{2}{27}$).

Since the magician takes 10-15 jumps through the deck, the sum of these probabilities is rather high; that is, $Ex(T)$ is around 3 or so. Consequently, Theorem 0.1 suggests that the magician is likely to eventually jump on a card visited by the volunteer. (As a ball-park estimate, the probability might be around $1-e^{-3}$.) In this case, the magician and volunteer end up with the same final card and the trick works.

Of course, Theorem 0.1 does not really apply, because the events A_i are not at all independent. For example, event A_i implies event A_{i+1}; once the magician and volunteer match, they always agree. This means that our probability estimates are bogus. However, computer tests suggest that for a full deck with jokers, the trick works more than 90% of the time.

Problem 2 The U.S. Supreme Court has mandated a re-vote in Florida. All 6,000,000 people who voted earlier will get to vote again. They are going to vote randomly, using a strategy that guarantees that the expected value of the number of votes for Bush is exactly 3,000,000. We don't know what the strategy is, though. (For example,
1. Each voter might toss a fair coin independently.
2. The Florida Supreme Court may toss one fair coin, and give all the votes to one candidate or the other based on the result.
3. 1/4 of the voters decide to vote for Bush, 1/4 for Gore, and the other half toss independent fair coins.
Etc.)

(a) What do Markov bounds tell us about the probability that Bush gets at least 6,000 more votes than Gore?

(b) Now assume also that the standard deviation of the experiment is 2000. What do Chebyshev bounds tell us about the probability that Bush gets at least 6,000 more votes than Gore?

(c) Now assume that we know the experiment. Specifically, one quarter of the voters decide ahead of time on Gore, one quarter decide on Bush, and the other half flip fair coins, independently. (This doesn't have a standard deviation of 2000.) What do Chernoff bounds tell us about the probability that Bush gets at least 6,000 more votes than Gore?

Problem 3 *The Probabilistic Method*

A round robin tournament of n contestants is one in which each of the $\binom{n}{2}$ pairs of contestants play each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. For a fixed integer k, $k < n$, a question of interest is whether it is possible that the tournament outcome is such that for every set of k players there is a player who beat each player of this set. Show that if:

$$\binom{n}{k}\left[1-\left(\frac{1}{2}\right)^k\right]^{n-k} < 1$$

then such an outcome is possible.

(a) Start by numbering the sets of k contestants. How many such sets are there?

(b) Let B_i be the event that no contestant beat all the k contestants in set i. Compute $\Pr(B_i)$. (Note that you must choose probabilities for each match in order to compute this).

(c) Give an upper bound on $\Pr(\cup B_i)$.

(d) Explain why this result can be used to prove the existence of the desired tournament outcome.