Problem Set 3

Due: 2:35pm in lecture, September 26, 2000.

Be sure to clearly indicate your TA's name on the solutions your turn in.

Self-study: More set theory Read: Velleman, Sections 1.3, 1.4, 2.3.
Self-study problems (not to be handed in): Velleman §1.3, Exercise 4, §1.4 Exercises 1, 3, 4, 5, 6, 8, §2.3 Exercises 2, 3, 5, 6

Reading: Velleman, Chapter 4
Rosen, Sections 3.2, 3.3, 3.4, Chapter 6.

Problem 1 Well-Ordering A set is **well-ordered** if every nonempty subset of the set has a least element. Determine whether each of the following sets is well-ordered. Explain your reasons.

(a) The set of integers.

(b) The set of integers greater than -100.

(c) The set of positive rationals.

(d) The set of positive rationals with a denominator less than 100, and a positive numerator.

Solutions:

(a) No. The set of integers is not well ordered. Consider the subset consisting of the entire set itself. This set has no smallest member, since there does not exist a smallest integer.

(b) Yes. This set is well-ordered. Observe that given a non-empty subset A, we can find its least element as follows. First, if $-99 \in A$, then this must be the least element since we are dealing with all integers greater than -100. If not, then we check if -98 is in the set A. If it is, then it must be the least element. If not, then we continue on to check for -97, and so on. Since A is non-empty, if we continue with this procedure, we will find the smallest element of A.

(c) No. The subset consisting of the entire set has no smallest element, since, given what might be "the smallest rational number," one could always divide that by two to get an even smaller rational number.

(d) Yes. Analogously to part (b), this set is well-ordered.

We prove this more formally by contradiction. Suppose that the set is not well ordered. Then, there is a subset A containing no least element. Let $\alpha_1 \in A$. If A has no least element, then there is some $\alpha_2 \in A$ with $\alpha_2 < \alpha_1$. Again, since α_2 cannot be the least element, there must be some $\alpha_3 \in A$ with $\alpha_3 < \alpha_2$. Repeating this process, we can come up with a never ending sequence of elements:

$$\alpha_1 < \alpha_2 < \alpha_3 < \ldots$$

Now, it follows, by definition, that α_1 can be written in the form m_1/n_1 where m_1 and n_1 are positive integers, and $n_1 < 100$. Similarly, α_2 can be written as m_2/n_2, and in general, α_i can be written as m_i/n_i. Now, if $\alpha_{i+1} < \alpha_i$, then it must be the case that $m_{i+1} < m_i$ or $n_{i+1} > n_i$ – for if it was the case that $m_{i+1} \geq m_i$ *and* $n_{i+1} \leq n = i$, then it would be the case that $\alpha_{i+1} \geq \alpha_i$. Also, by definition, each of the n_i must be less than 100. Thus at some point i, n_{j+1} cannot be greater than n_j, for all $j \geq i$. From that point onward, m_{j+1} must be strictly less than m_j, m_{j+2} must be strictly less than m_{j+1}, and so on. However, since each m_j is an integer, at some point this sequence of values m_j will become negative. This contradicts our original assumption that the numerators are positive.

Problem 2 Structural Induction

The set SSS of "super-symmetric strings" is defined as follows:

1. The length one strings consisting of individual letters of the alphabet, a, b, c, d, $\ldots$, z, are in SSS.
2. If α and α' are two strings in SSS then the string $\alpha\alpha'\alpha$ is in SSS.
3. SSS contains nothing else.

(a) Which of the following are super-symmetric strings?

i. a
ii. aaaba
iii. abcbacabcba
iv. λ, the empty string
v. abaabcbaaba

Solutions:

i. Yes. This is stated in part 1 of the definition.
ii. No. This string is not of the form $\alpha\alpha'\alpha$.
iii. Yes. Formally, consider $\alpha' = aca$, $\alpha = bcb$. Then we have a string of the form $a\alpha\alpha'\alpha a$, which is in SSS by parts 1 and 2 of the definition.
iv. No. By part 3 of the definition
v. Yes. Similar reasoning to part iii shows that the string bcb is in the middle of the super-symmetric string, with the string a wrapped around it, and with the string aba wrapped around that.

(b) Prove by structural induction that in any super-symmetric string α, exactly one letter appears an odd number of times.

Solution:
Proof: Define $P(e)$: String e has exactly one letter which appears an odd number of times.

Prove: $\forall e \in SSS\ P(e)$.
1. (Base) $P(a)$ The string a has exactly one letter which appears an odd number of times.
2. (Base) $P(b)$ Same for the string b

$\vdots$

26. (Base) $P(z)$ Still has exactly one letter which appears once.
27. (Inductive Step) $\forall e, e' \in SSS\ P(e) \land P(e') \longrightarrow P(ee'e)$
    1. Fix $e, e' \in SSS$.
    2. Assume $P(e) \land P(e')$. That is, each of e and e' has exactly one letter which appears and odd number of times.
    3. By the inductive hypothesis, both e and e' have exactly one letter which appears an odd number of times. When we form the string $ee'e$, the letters in e all get repeated, and thus they all appear an even number of times. By the IH, there is one letter in e' which appears an odd number of times in e'. Now, even if this letter is also present in e, it will still appear an odd number of times in $ee'e$ (since the sum of an odd number and an even number is odd). Therefore, we have shown $P(ee'e)$.
    4. QED.

Problem 3 Induction and Recursive Algorithms

(a) Rosen §3.4, Exercise 18
Devise a recursive algorithm to find the nth term of the sequence defined by $a_0 = 1, a_1 = 2, a_2 = 3$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, for $n = 3, 4, 5, \ldots$.

Solution:

procedure *sum-sequence* (n:nonnegative integers)
if (n<3) **then** *sum-sequence*(n):=$n+1$;
else *sum-sequence*(n):=*sum-sequence*$(n-1)$+ *sum-sequence*$(n-2)$+*sum-sequence*$(n-3)$

Problem 4 Relations

(a) Velleman §4.2, Exercise 3 parts a,b and c.
Suppose $A = \{1,2,3\}$, $B = \{4,5,6\}$, $R = \{(1,4),(1,5),(2,5),(3,6)\}$, and $S = \{(4,5),(4,6),(5,4),(6,6)\}$. Note that R is a relation from A to B and S is a relation from B to B. Find the following relations:
a. $S \circ R$

b. $S \circ S$
c. $S^{-1} \circ R$

Suppose $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, $R = \{(1, 4), (1, 5), (2, 5), (3, 6)\}$, and $S = \{(4, 5), (4, 6), (5, 4), (6, 6)\}$. Note that R is a relation from A to B and S is a relation from B to B. Find the following relations:

Solutions:
a. $S \circ R = \{(1, 5), (1, 6), (1, 4), (2, 4), (3, 6)\}$
b. $S \circ S = \{(4, 4), (4, 6), (5, 5), (5, 6), (6, 6)\}$
c. $S^{-1} \circ R = \{(1, 5), (1, 4), (2, 4), (3, 4), (3, 6)\}$

Problem 5 Relations — Properties

(a) Velleman §4.4, Exercise 1.

In each case, say whether or not R is a partial order on A. If so, is it a total order?
a. $A = \{a, b, c\}$, $R = \{(a, a), (b, a), (b, b), (b, c), (c, c)\}$.
b. $A = \{$ the set of all words in English, $R = \{(x, y\} \in A \times A \mid$ the word x comes before the word y alphabetically$\}$.
c. $A = \mathbb{R}, R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x| \leq |y|\}$.
d. $A = \mathbb{R}, R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid |x| < |y| \text{ or } x = y\}$.

Solutions:

a. R is reflexive, transitive, and antisymmetric. Therefore, R is a partial order on A. However, it is not a total order on A since neither the pair (a, c) nor (c, a) is in R.
b. R is not reflexive (although it is transitive and antisymmetric), and thus it is not a partial order on A.
c. R is reflexive, transitive, and antisymmetric. Therefore, R is a partial order on A. It is also a total order on A, since $\forall x, y \in A$, it is true that either $|x| \leq |y|$, or $|y| \leq |x|$.
d. R is reflexive, transitive, and antisymmetric. Therefore, R is a partial order on A. Also, R is a total order by an analysis similar to part c.

Problem 6 Relations — Equivalence relations

(a) Velleman §4.6, Exercise 3.

Which of the following relations are equivalence relations? For those that are equivalence relations, what are the equivalence classes?

a. $R = \{(x, y) \in W \times W \mid$ the words x and y start with the same letter$\}$,
where $W =$ the set of all words in teh English language.
b. $S = \{(x, y) \in W \times W \mid$ the words x and y have at least one letter in common$\}$, where $W =$ the set of all words in teh English language.
c. $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x - y \in \mathbb{Q}\}$.
d. $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid \exists n \in \mathbb{Z} \text{ such that } (y = x \cdot 10^n)\}$.

Solutions:

a. R is an equivalence relation since it is reflexive, symmetric, and transitive. The equivalence class of x with respect to R is the set $[x]_R$ = the set of words y, such that y has the same first letter as x.

b. S is reflexive and symmetric, but it is not transitive. Therefore, S is not an equivalence relation. For example, let w_1 be the word "scream," let w_2 be the word "and," and let w_3 be the word "shout." Then $w_3 S w_1$, and $w_1 S w_2$, but it is **not** the case that $w_3 S w_2$.

c. R is reflexive since, for any $x \in R$, $x - x = 0$. R is symmetric because if $(x - y)$ is rational, then $(y - x) = -(x - y)$ is rational as well. Finally, R is also transitive. To see this, consider $(x, y) \in R$ and $(y, z) \in R$. We would like to know if (x, z) is also in R. We start by noting that $(x - y) \in \mathbb{Q}$ and $(y - z) \in \mathbb{Q}$ by definition of our relation R. Now, we can express $(x - z) = (x - y) + (y - z)$. Since the sum of any two rational numbers is another rational number, and we have written $(x - z)$ as such, we have shown that $(x, z) \in \mathbb{Q}$. Therefore, R is transitive. Since R is also reflexive and symmetric, R is an equivalence relations on $\mathbb{R}$.
The equivalence class for x is $[x]_R = \{y \mid y$ can be written in the form $x + r$, for some rational number $r\}$

d. R is reflexive, since we can set $n = 0$. R is symmetric because if $y = x * 10^k$, for some $k \in \mathbb{Z}$, then $x = y * 10^{-k}$, so if $(x, y) \in R$, then $(y, x) \in R$. We can show that R is transitive as follows: consider pairs $(x, y) \in R$ and $(y, z) \in R$. $y = x * 10^n$ and $z = y * 10^m$, for $m, n \in \mathbb{Z}$. Then, substituting, we have that $z = y * 10^{m+n}$. Since $m + n$ is also $\in \mathbb{Z}$, (x, z) is $\in R$.
The equivalence class for x is $[x]_R = \{y \mid y = x * 10^n$, for some $n \in \mathbb{Z}\}$

(b) Velleman §4.6, Exercise 5.
Given T, the set of all triangles, and $S = \{(s, t) \in T \times T \mid$ the triangles s and t are similar$\}$, prove that S is an equivalence relation.

Solution:

Proof: To prove this, we need to show that S is reflexive, symmetric, and transitive. Here we go:
First of all, two triangles s and t are said to be *similar* if they have three pairs of equal-degree vertices.
S is reflexive because every triangle is similar to itself. That is, every triangle s has three angles of equivalent size to its own three angles.
S is symmetric because if s is similar to t, then s and t have vertices of the same size, so t is similar to s as well.
Finally, S is transitive because if triangle s is similar to triangle t, and triangle t is similar to triangle r then triangles s and t have three pairs of equal-degree vertices, and triangles t and r also have three pairs of equal-degree vertices. s and r must share these three pairs of equal-degree vertices since triangles only have three vertices. $\blacksquare$.