Problem Set 8.5 Solutions

You do not have to turn in the solutions to this Problem Set. It is designed to help you study for the second quiz.

Reading: Lecture notes 17-18.

Quiz 2: Wednesday, November 15 7-9pm. Covers lectures 11-18.

Problem 1 **Vote today** $6N$ voters show up to vote for one of three candidates: Bush, Gore and Nader.

(a) Assuming that voters are indistinguishable, what is the number of possible vote counts?

Solution: We can model this problem using $6N$ stars, each representing a vote, and two bars, to separate Gore's, Bush's and Nader's votes. Each election result corresponds to a unique placement of the stars and bars. Therefore, the number of possible election results is $\binom{6N+2}{2}$.

(b) What is the number of possible vote counts in which Nader wins the popular vote?

Solution: Let T be the number of vote counts for which the election is a tied race. By symmetry, Nader can win the election in $\frac{1}{3}(\binom{6N+2}{2} - T)$ ways.

To compute T, we use a similar trick. A tie happens either if two candidates tie for first place or all three candidates tie for first place. By symmetry, we only need to consider the case when Bush and Gore tie for first (but Nader doesn't). For this to happen, Bush and Gore must each get at least $2N+1$ votes, but no more than $3N$ votes. This is a total of N possible outcomes. It follows that $T = 3N+1$, as there is one way in which all candidates tie for first.

It follows that Nader can win the election in $\frac{1}{3}(\binom{6N+2}{2} - 3N - 1) = 2N(3N+1)$ ways.

Problem 2 How many ways are there to distribute eight balls into six distinct boxes with the first two boxes having collectively at most four balls if

(a) the balls are identical?

Solution: Let us consider placing n balls into k boxes so that exactly r balls are placed in the first two boxes. This can be done in

$$\binom{r+2-1}{2-1}\binom{(n-r)+(k-2)-1}{(k-2)-1}$$

ways, since there are $\mathrm{BP}(r,2) = \binom{r+2-1}{2-1}$ ways to place r balls into first two boxes and $\mathrm{BP}(n-r, k-2) = \binom{(n-r)+(k-2)-1}{(k-2)-1}$ ways to place the remaining $n-r$ balls into the other $k-2$ boxes. We can place the balls into boxes, so that *at most* r balls go into the first two boxes, in

$$\sum_{i=0}^{r} \binom{i+2-1}{2-1} \binom{(n-i)+(k-2)-1}{(k-2)-1}$$

ways. In particular for $n = 8$, $k = 6$ and $r = 4$ we get

$$\sum_{i=0}^{4} \binom{i+2-1}{2-1} \binom{(8-i)+(6-2)-1}{(6-2)-1} = 1056.$$

(b) the balls are distinct?

Solution: As in the part (a), we compute the number of ways n balls can be placed into k boxes so that exactly r of them are placed in the first two boxes. Since the balls are distinct, we have $\binom{n}{r}$ ways to choose which balls to place in the first two boxes, and then we have 2^r ways to place the balls into first two boxes, and $(k-2)^{n-r}$ ways to place the rest of the balls into the remaining $k-2$ boxes. Thus the total number of posibilities is

$$\binom{n}{r} 2^r (k-2)^{n-r}.$$

If we require that *at most* r balls go into the first two boxes we compute that the number of posibilities is

$$\sum_{i=0}^{r} \binom{n}{i} 2^i (k-2)^{n-i}.$$

For $n = 8$, $k = 6$ and $r = 4$ we get

$$\sum_{i=0}^{4} \binom{8}{i} 2^i (6-2)^{8-i} = 1531904.$$

Problem 3 How many ways can you choose n out of $2n$ objects, given that n of the $2n$ objects are identical?

Solution: The answer is 2^n, since you can pick any subset from the n nonidentical objects, and make up the rest with the identical ones. And the number of subsets of n different objects is 2^n.

Problem 4 How many different ways are there to list the numbers $1, 2, \ldots, 2n$ such that the even numbers appear in increasing order and the odd numbers appear in decreasing order?

Solution: There are $(2n)!$ ways to list the numbers $1, 2, \ldots, n$. We can define a $n!$-to-1 mapping that takes an arbitrary listing to a listing in which the even numbers are sorted in increasing order and the odd numbers remain as before. Thus, by the quotient rule there are $(2n)!/n!$ listings in which the even numbers appear in increasing order. Similarly, we can define an $n!$-to-1 mapping that takes a listing in which the even numbers appear in increasing order and the odd numbers are in an arbitrary order to a listing in which the even numbers appear in increasing order (as before) and the odd numbers appear in decreasing order. Therefore, there are

$$\frac{(2n)!}{n!\,n!} = \binom{2n}{n}$$

listings in which the even numbers are increasing and the odd numbers are decreasing.

Problem 5 We wish to prove the identity

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n},$$

for $n \in \mathbb{N}$.

(a) Use the binomial theorem and the fact that

$$(1+x)^n(1+x)^n = (1+x)^{2n}$$

to prove the identity.

Solution:

$$\begin{aligned} (1+x)^n(1+x)^n &= (1+x)^{2n} \\ \sum_{i=0}^{n} \binom{n}{i} x^i \sum_{j=0}^{n} \binom{n}{j} x^j &= \sum_{i=0}^{2n} \binom{2n}{i} x^i \end{aligned}$$

Equating the coefficients of x^n on both sides yields

$$\begin{aligned} \sum_{i+j=n} \binom{n}{i}\binom{n}{j} &= \binom{2n}{n} \\ \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} &= \binom{2n}{n} \end{aligned}$$

From where we obtain the identity using the substitution $\binom{n}{n-i} = \binom{n}{i}$.

(b) Consider a set of $2n$ objects, of which n are red and n are blue. Give a combinatorial proof of the identity.

Solution: Suppose we have $2n$ objects, n of which are red, and n of which are blue. How many ways can we pick r red ones and $n-r$ blue ones? Clearly this is

$$\binom{n}{r}\binom{n}{n-r} = \binom{n}{r}^2.$$

How many ways can we choose n objects total, regardless of how many are of each color? This is just

$$\sum_{r=0}^{n} \binom{n}{r}^2.$$

But there are $\binom{2n}{n}$ ways to choose n objects, so these must be the same number.

Problem 6 Prove or disprove the following statements about sets.

(a) The product of a finite number of countable sets is countable.

Solution: The statement is true. The proof is by induction. Let $P(n)$ be the proposition that the product of n countable sets is countable. In the base case, $P(1)$ is true because the product of a single countable set is itself. In the inductive step, for $n \geq 1$ assume $P(n)$ to prove $P(n+1)$. The product of $n+1$ countable sets $S_1, S_2, \ldots, S_{n+1}$ can be written as shown below.

$$(S_1 \times S_2 \times \ldots \times S_n) \quad \times \quad S_{n+1}$$

The first term is a countable set by induction, and the second term is a countable set by definition. The product of two countable sets is countable, as shown in lecture. Therefore, the product of $n+1$ countable sets is countable. This shows that $P(n)$ implies $P(n+1)$, and the claim is proved by induction.

(b) A countable union of countable sets is countable.

Solution: This statement is true. Let $\{S_i, i \in \mathbb{N}\}$ be a collection of countable sets. Write $S = \bigcup_{i \in \mathbb{N}} S_i$. Since each S_i is countable, there are bijections $f_i : \mathbb{N} \to S_i$ for all $i \in \mathbb{N}$. The map $f : \mathbb{N} \times \mathbb{N} \to S$ given by $f(i,j) = f_i(j)$ is surjective. Since $\mathbb{N} \times \mathbb{N}$ is a product of two countable sets, it is countable. It follows that S must be countable.

Problem 7 Use a diagonalization argument to show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Solution: Assume, for the purposes of contradiction, that this set is countable. Then there is a listing of the function in the set, $f_1, f_2, \ldots$. Now define f such that $f(i) = (f_i(i) + 5) \mod 10$. Then f is not in the list, because f differs from each f_i in the ith value. We have established a contradiction. We conclude that the set is uncountable.