Practice Quiz 2

Problem 1 [10 points]

Consider $n > 1$ people that attend a party. Prove that there must be two people who greet the same number of distinct people; assuming no one greets his/her self.

Problem 2 [20 points] **Counting**

For the problems below, you may leave products and binomial coefficients in your final answer. Moreover, you should assume that a deck of cards consists of 52 cards; that is, 4 suits of 13 cards each $(2, 3, 4, \ldots, 9, 10, J, Q, K, A)$.

(a) [5 points] In poker, a *flush* is a set of five cards, all of the same suit. How many distinct flushes are there?

(b) [5 points] In poker, a *straight* is a set of five cards that, when the suits are ignored, form a five-in-a-row sequence, for example $5, 6, 7, 8, 9$ or $8, 9, 10, J, Q$. An ace can start a straight $(A, 2, 3, 4, 5)$ or end it $(10, J, Q, K, A)$ but cannot wrap around (**no** $Q, K, A, 2, 3$). How many distinct straights are there?

(c) [5 points] A *straight flush* is a set of five cards that is both a straight and a flush: 5 cards in a row, all the same suit. How many are there?

(d) [5 points] How many 5-card sets are either a straight *or* a flush (or both)?

Problem 3 [10 points] **Counting**

Consider placing n distinguishable books on r distinguishable shelves.

(a) [5 points] If the order of books on each shelf doesn't matter, how many distinct placements are there?

(b) [5 points] If order on the shelf matters, how many distinct placements are there?

Problem 4 [10 points] **Order of Growth**

(a) [3 points] From lecture, we know that any comparison-based sorting algorithm $\mathbb{A}$ must take $\Omega(n \log n)$ steps on n elements. Which one of the following is a consequence of this fact?

a) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform exactly $cn \log n$ steps.

b) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform at least $cn \log n$ steps.

c) There is some constant $c > 0$ for which some executions of $\mathbb{A}$ will perform at least $cn \log n$ steps.

d) There is some constant $c > 0$ for which all executions of $\mathbb{A}$ will perform at most $cn \log n$ steps.

e) There is some constant $c > 0$ for which some executions of $\mathbb{A}$ will perform at most $cn \log n$ steps.

(b) [3 points] Prove that $10n^2 = O(n^3)$.

(c) [4 points] Prove that n^3 is not $O(10n^2)$.

Problem 5 [10 points] **Summations**

(a) [5 points] Determine the closed-form expression for the summation:

$$\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} x^{k+j} y^{k-j}$$

(b) [5 points] Prove a tight asymptotic bound (Θ) for the expression:

$$\sum_{k=2}^{n} \frac{1}{k \ln k}$$

Hint: $\dfrac{1}{x \ln x} = \dfrac{d}{dx} \ln \ln x$

Problem 6 [10 points] **Binomial Coefficients**
Prove that

$$\sum_{k=1}^{n} k \binom{n}{k} = n 2^{n-1}$$

(a) [5 points] using a combinatorial argument. **Hint:** Consider choosing a committee that is headed by a leader.

(b) [5 points] by finding a closed-form expression for $\sum_{k=1}^{n} \binom{n}{k} kx^k$ and evaluating it at $x = 1$.

Problem 7 [10 points] **Recurrences**

Consider the following recurrence:

$$T(n) = 2T(n/2) + n \log_2 n$$

(a) [3 points] Perform two iterations of plug-and-chug to reveal a pattern for this recurrence.

(b) [7 points] Determine a tight asymptotic bound (Θ) for this recurrence. You may assume n is a power of two if it is convenient.

Problem 8 [10 points] **Recurrences**

The Millicent computer virus has been spreading on computers around MIT. Once a copy of Millicent enters a computer, it is copied to 2 new (uninfected) computers after one day, then to 3 new computers the second day, and then detected and removed on the third day. Millicent's creator installed Millicent on 2 computers on day 0 and 2 additional computers on day 1.

(a) [3 points] Due to a tip from inside the FBI that Millicent might be of extraterrestrial origin, Frohike and Langley have been investigating the growth of the Millicent virus; however, they've gotten stuck. Help them by stating a recurrence describing the number of computers **newly** infected on day n for $n > 1$.

(b) [4 points] Find a closed form for the recurrence you stated in the previous part.

(c) [3 points] Fearing the efforts of the mighty Frohike and suave Langley, Millicent's creator became frightened and decided to start installing the virus on 4 new computers every day after day 1. State a recurrence for the revised problem and find a closed form solution for the number of computers newly infected on day n.

Problem 9 [10 points] **Countability**

(a) [5 points] A function $f : \mathbb{N} \to \mathbb{N}$ is *strictly increasing* if $f(n+1) > f(n)$ for all n. Prove that the set of such strictly increasing functions is uncountable. **Hint:** Use diagonalization.

(b) [5 points] A function $f : \mathbb{N} \to \mathbb{N}$ is *strictly decreasing* if $f(n+1) < f(n)$, for all n such that $f(n) > 0$, and $f(n+1) = 0$, for all n such that $f(n) = 0$. For example, the function f could be $f(0) = 5, f(1) = 3, f(2) = 0, f(3) = 0, \cdots$.

Prove that the set of such strictly decreasing functions is countable. **Hint:** First prove that, for any $k \in \mathbb{N}$, the set of strictly decreasing functions for which $f(0) = k$ is countable.